The Proofwriting Workbook

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Contents

Chapter 1

Foundations

1.1 Statements and Truth

Definition 1.1.1. *A **statement** or **proposition** is a sentence that we can determine must be either true or false.*

1. Which of the following are statements? If they are statements, what is the their truth value?

   (a) Take the derivative of $f(x)$. [Answer: Not a statement. This sentence is a command and not something that can really be considered true or false. Maybe you will, or wont take the derivative if someone tells you do, but the request itself has no truth value.]

   (b) The derivative of $\sin(x)$. [Answer: Not a statement. This isn't even a full sentence.]

   (c) The derivative of $\cos(x)$ is equal to negative $\sin(x)$. [Answer: True statement.]

   (d) The sum of $\sin(x)$ and $\cos(x)$ is always equal to one, for any number x. [Answer: False statement.]

   (e) The sum of $\sin(x)$ and $\cos(x)$. [Answer: Not a statement. This is not even a full sentence.]

   (f) Is the derivative of $\ln(x)$ equal to $\frac{1}{x}$? [Answer: Not a statement. This is a question.]

   (g) The derivative of $\frac{1}{x}$ is $\ln(x)$. [Answer: False statement.]

   (h) Add two even numbers. [Answer: Not a statement. This is again a command.]

   (i) What is the sum of an even and an odd number? [Answer: Not a statement.]

   (j) If you add an even and an odd number you will get an even number. [Answer: False statement.]

   (k) The smallest prime number is one. [Answer: False statement.]

   (l) The smallest prime number less than one hundred. [Answer: Not a statement. Not a complete sentence in fact.]

   (m) The derivative of $\sin^2(x)$ is the same as the derivative of $-\cos^2(x)$. [Answer: True statement.]

   (n) The product of two nonzero numbers is a nonzero number. [Answer: True statement. The only way a product is zero is if one of the two terms is zero.]

   (o) The sum of two nonzero numbers is a nonzero number. [Answer: False statement. Sometimes the sum of a positive and a negative gives you zero as the result.]

Definition 1.1.2. *A logical operator is a symbol used to connect propositions to form a new sentence whose truth value depends only on the values of those proposition in some way.*

Definition 1.1.3. *The negation of a statement P is the new statement, written $\sim P$, which is considered true if P is false and false if P is true.*

Definition 1.1.4. *The conjunction of two statements P and Q is the new statement, written $P \wedge Q$, which is considered true only if both P and Q are true, and false otherwise.*

Definition 1.1.5. *The disjunction of two statements P and Q is the new statement, written $P \vee Q$, which is considered true if at least one of P and Q are true, and false otherwise.*

2. Let P be the statement "three is bigger than four" and Q be the statement "five is bigger than four." Convert the following into normal English sentences and state which of the following are true. There are many ways to do these conversions, though the truth value will always be the same.
   (a) $\sim Q$ [Answer: "Five is less than or equal to four." False.]
   (b) $\sim (\sim Q)$ [Answer: "Five is not less than or equal to four." True. One could also say "five is not not bigger than four" or "five is bigger than four."]
   (c) $P \wedge Q$ [Answer: "Three and five are both bigger than four." False.]
   (d) $P \vee Q$ [Answer: "Three or five is bigger than four." True.]
   (e) $(\sim P) \vee (\sim Q)$ [Answer: "Three or five is less than or equal to four." True.]
   (f) $(\sim P) \wedge (\sim Q)$ [Answer: "Neither three nor five is bigger than four." False. We could also write "three or five is less than or equal to four."]

3. Find statements P and Q that do not contain the words "and", "or", and "not" are use them to rewrite the following sentences without those words. Use only the symbols $P, Q, \sim, \wedge$, and $\vee$ to accomplish this.
   (a) Four is positive and even. [Answer: $P =$"four is positive", $Q =$ "four is even", $P \wedge Q$.]
   (b) The number n is even or odd.
   (c) Four is not prime and neither is six. [Answer: $P =$"four is prime", $Q =$ "six is prime", $(\sim P) \wedge (\sim Q)$.]
   (d) Four is not prime but five is. [Answer: $P =$"four is prime", $Q =$ "five is prime", $(\sim P) \wedge Q$.]
   (e) Nine is odd and not a prime number.
   (f) Zero is even and not positive.
   (g) Zero is neither positive nor negative. [Answer: $P =$"zero is positive", $Q =$ "zero is negative", $(\sim P) \wedge (\sim Q)$.]

Definition 1.1.6. *The truth table for a statement Q involving propositions $P_1, P_2, \cdots, P_n$, together with some arrangement of logical operators, is a table containing every possible arrangement of truth values for our P_i together with whether or not Q is true or false under those conditions.*

4. Write out truth tables for the following:

(a) $\sim P$

Answer: We make a truth table with two columns. The first contains both possible truth values for P. The second column contains the values for $\sim P$ given that we know $\sim P$ is true if and only if P is false.

P	$\sim P$
T	F
F	T

(b) $P \wedge Q$

Answer: Here we will need more rows, since we must list every possible combination of truth values for both P and Q. The entries in the final column contain the truth value of $P \wedge Q$ given the truth values of P and Q, which are listed previously in that row. By the definition of $P \wedge Q$ the only row to have a T in the rightmost column must correspond to the row where both P and Q are true.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

(c) $P \vee Q$

Answer:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

(d) $P \wedge \sim Q$

Answer: So far we've only constructed truth tables for the three logical operators we've examined individually. To efficiently make truth tables when there are multiple operators in our sentence, we can use the fact that all three operators each rely on at most two propositions. This allows us to break up what might be a confusing sentence into parts and compute each new column using at most two previous ones.

For example, here we can first make columns for P and Q and then use the Q column to construct the $\sim Q$ column. We can then ignore the column for Q while we use the P and and $\sim Q$ columns to construct the column for $P \wedge \sim Q$.

P	Q	$\sim Q$	$P \wedge \sim Q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

(e) $P \wedge (Q \vee P)$

Answer: Here we want to make and use columns for P and $Q \vee P$ to find the values for $P \wedge (Q \vee P)$. We will use the column for Q to find the values for $Q \vee P$ but ignore that column when computing the values in the last step.

P	Q	$Q \vee P$	$P \wedge (Q \vee P)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

(f) $P \vee (Q \wedge \sim P)$

Answer:

P	Q	$\sim P$	$Q \wedge \sim P$	$P \vee (Q \wedge \sim P)$
T	T	F	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	F	F

(g) $(P \wedge Q) \vee (\sim Q)$

(h) $P \wedge (\sim Q \vee Q)$

(i) $(\sim P \wedge \sim Q) \vee P$

Answer:

P	Q	$\sim P$	$\sim Q$	$(\sim P \wedge \sim Q)$	$(\sim P \wedge \sim Q) \vee P$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	F
F	F	T	T	T	T

(j) $\sim (P \wedge (\sim Q \vee Q))$

(k) $\sim P \wedge (\sim Q \vee Q)$

(l) $P \vee (Q \wedge R)$

Answer: The columns need to get longer to accommodate the three sentences P, Q, and R. Aside from the fact that we have eight starting possibilities the strategy is basically the same.

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

(m) $\sim ((P \vee Q) \vee R)$

P	Q	R	$P \vee Q$	$(P \vee Q) \vee R$	$\sim ((P \vee Q) \vee R)$
T	T	T	T	T	F
T	T	F	T	T	F
T	F	T	T	T	F
T	F	F	T	T	F
F	T	T	T	T	F
F	T	F	T	T	F
F	F	T	F	T	F
F	F	F	F	F	T

(n) $(\sim P \wedge Q) \wedge \sim R$

(o) $\sim P \wedge \sim (Q \vee R)$

Definition 1.1.7. *A statement made from the propositions $P_1, P_2, \cdots, P_n$ is a tautology if it is true for every possible assignment of truth values for our P_i.*

Definition 1.1.8. *A statement made from the propositions $P_1, P_2, \cdots, P_n$ is a contradiction if it is false for every possible assignment of truth values for our P_i.*

5. Figure out which of the following statements are tautologies, contradictions, or neither. You can use a truth table if you like, but you may not need to.

(a) $P \vee \sim P$ [Answer: Tautology. The only way a disjunction can be false is if both parts are false. This can not happen since whenever P is false, $\sim P$ is true. One can also easily make a truth table to confirm this as follows:

P	$\sim P$	$P \vee \sim P$
T	F	T
F	T	T

]

(b) $P \wedge \sim P$ [Answer: Contradiction. The only way a conjunction can be true is if both parts are true. This can not happen since whenever P is true, $\sim P$ is not.]

(c) $(P \vee Q) \vee \sim P$ [Answer: Tautology. The only way the outermost disjunction can be false is if both parts are false. If $\sim P$ were false then $P \vee Q$ would be true because P would be true. Thus, both parts can't be false and the disjunction is always true.]

(d) $(P \wedge Q) \wedge \sim P$ [Answer: Contradiction. The only way the outermost conjuction can be true is if both parts are true. If the first part, $P \wedge Q$, were true then P would have to be true. That would mean the second part, $\sim P$, would be false.]

(e) $(P \vee Q) \wedge \sim P$ [Answer: Neither. If P were false and Q were true, then both parts of the outer conjunction would be true, so the statement would be true. If both P and Q were true, the second part and thus the entire statement would be false. Thus it is possible for the outcome to be either true or false, depending on the values of P and Q. We can also make a truth table as follows:

P	Q	$P \vee Q$	$\sim P$	$(P \vee Q) \wedge \sim P$
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

]

(f) $(P \wedge Q) \vee ((\sim P) \vee (\sim Q))$

(g) $(P \wedge Q) \wedge ((\sim P) \vee (\sim Q))$

Definition 1.1.9. *Two statements Q and R, made from the propositions $P_1, P_2, \cdots, P_n$ are logically equivalent if they are both true under the exact same conditions of the truth values of our P_i. In this case we write $P \equiv R$.*

6. Use truth tables to show the following logical equivalences:

   (a) Idempotent Laws:

      i. $P \equiv P \wedge P$

      ii. $P \equiv P \vee P$

   (b) Double Negation Law:
   $P \equiv \sim (\sim P)$

   (c) Commutativity Laws:

      i. $P \vee Q \equiv Q \vee P$

      ii. $P \wedge Q \equiv Q \wedge P$

   (d) Associativity Laws:

      i. $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$
      Answer:

P	Q	R	$Q \vee R$	$P \vee (Q \vee R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	F

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P	Q	R	$P \vee Q$	$(P \vee Q) \vee R$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	T
F	F	F	F	F

      ii. $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$

   (e) Distributivity Laws:

      i. $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
      Answer:

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

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P	Q	R	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F
F	F	F	F	F	F

ii. $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

(f) DeMorgan's Laws:

i. $\sim (P \wedge Q) \equiv (\sim P) \vee (\sim Q)$

ii. $\sim (P \vee Q) \equiv (\sim P) \wedge (\sim Q)$

(g) Absorption Laws:

i. $P \equiv P \vee (P \wedge Q)$

Answer:

P		P	Q	$P \wedge Q$	$P \vee (P \wedge Q)$
T		T	T	T	T
T	,	T	F	F	T
F		F	T	F	F
F		F	F	F	F

ii. $P \equiv P \wedge (P \vee Q)$

1.2 Conditionals and Biconditionals

1. Make truth tables for the following statements.

   (a) $P \wedge (P \Rightarrow P)$

   (b) $(P \Rightarrow Q) \vee (Q \Rightarrow P)$

   (c) $(P \wedge Q) \Rightarrow P$

   Solution:

P	Q	$P \wedge Q$	$(P \wedge Q) \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

   (d) $(P \wedge Q) \Rightarrow (P \vee Q)$

   (e) $(P \vee Q) \Rightarrow (P \wedge Q)$

   (f) $P \Rightarrow (Q \wedge P)$

   (g) $(\sim P \Rightarrow Q) \wedge (P \Rightarrow \sim Q)$

   Solution:

P	Q	$\sim P$	$\sim P \Rightarrow Q$	$\sim Q$	$P \Rightarrow \sim Q$	$(\sim P \Rightarrow Q) \wedge (P \Rightarrow \sim Q)$
T	T	F	T	F	F	F
T	F	F	T	T	T	T
F	T	T	T	F	T	T
F	F	T	F	T	T	F

   (h) $(\sim P \Rightarrow Q) \wedge (Q \Rightarrow \sim P)$

   Solution:

P	Q	$\sim P$	$\sim P \Rightarrow Q$	$Q \Rightarrow \sim P$	$(\sim P \Rightarrow Q) \wedge (Q \Rightarrow \sim P)$
T	T	F	T	F	F
T	F	F	T	T	T
F	T	T	T	T	T
F	F	T	F	T	F

   (i) $(\sim P \Rightarrow Q) \vee (Q \Rightarrow \sim P)$

   (j) $(P \Rightarrow Q) \wedge (P \Rightarrow R)$

   (k) $(P \Rightarrow Q) \wedge (Q \Rightarrow R)$

   Solution:

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$(P \Rightarrow Q) \wedge (Q \Rightarrow R)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

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(l) $(P \Rightarrow Q) \vee (Q \Rightarrow R)$

2. Consider the following sentences:

   P : n is prime.

   S : n is square.

   E : n is even.

   Q : $n+1$ is prime.

   Rewrite the following statements using only these sentences and the symbols $\Rightarrow$ and $\sim$. Then state whether the statement is true or false. If it is false, then explain why. Recall that an integer is square if it is equal to some integer squared, and the successor of an integer is that integer plus one.

   (a) If n is a square then it is not prime.

   (b) If n is not prime then it is a square.

   (c) If n is not a square then it is prime.

   (d) If n is prime then it is not a square.

   (e) If n is not a square then it is not a prime.

   (f) If n is not a prime then it is not a square. [Answer: $\sim P \Rightarrow \sim S$, False. For example, the number four is not a prime yet it is a square.]

   (g) If n is a prime then it is not even.

   (h) If n is not prime then it is even. [Answer: $\sim P \Rightarrow E$, False. For example the number nine is not a prime and it is not even.]

   (i) If n is not prime then it is not even.

   (j) If n is not even then it is not prime.

   (k) If n is prime then its successor is prime. [Answer: $P \Rightarrow Q$, False. For example three is prime, but its successor four is not.]

   (l) If n is prime then its successor is not prime. [Answer: $P \Rightarrow \sim Q$, False. For example two is prime, but its successor three is also prime.]

   (m) If n is not prime then its successor is prime. [Answer: $\sim P \Rightarrow Q$, False. For example, eight is not prime but its successor is also not prime.]

   (n) If n is not prime then its successor is not prime.

3. Consider the following sentences about an arbitrary integer n:

   A : n is a natural number.

   B : n^2 is a natural number.

   C : n is a negative number.

   D : n^2 is a negative number.

   Translate the following into english sentences and state whether they are true or false. Use that the natural numbers are the whole positive numbers $1, 2, 3, \cdots$ and the integers are all the whole numbers $\cdots, -2, -1, 0, 1, 2, \cdots$.

(a) $A \Rightarrow B$ [Answer: The square of a natural number is natural. True.]
(b) $B \Rightarrow A$ [Answer: If n^2 is natural, so is n. False.]
(c) $C \Rightarrow B$
(d) $B \Rightarrow C$
(e) $B \Rightarrow A \vee C$
(f) $C \Rightarrow D$
(g) $D \Rightarrow A$
(h) $D \Rightarrow B$
(i) $D \Rightarrow C$ [Answer: If n^2 is negative, so is n. True.]

4. Use truth tables to prove the contrapositive law: $P \Rightarrow Q \equiv \sim Q \Rightarrow \sim P$.

5. Use truth tables to prove the law of material implication: $P \Rightarrow Q \equiv \sim P \vee Q$

6. Use truth tables to show the following logical equivalences for conditional statements.

(a) $P \vee Q \equiv \sim P \Rightarrow Q$
(b) $P \wedge Q \equiv \sim (P \Rightarrow \sim Q)$
(c) $P \wedge \sim Q \equiv \sim (P \Rightarrow Q)$
(d) $P \Rightarrow (Q \Rightarrow R) \equiv Q \Rightarrow (P \Rightarrow R)$
Solution:

P	Q	R	$Q \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

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P	Q	R	$P \Rightarrow R$	$Q \Rightarrow (P \Rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

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(e) $(P \Rightarrow Q) \wedge (P \Rightarrow R) \equiv P \Rightarrow (Q \wedge R)$
(f) $(P \Rightarrow Q) \vee (P \Rightarrow R) \equiv P \Rightarrow (Q \vee R)$
(g) $(P \Rightarrow R) \wedge (Q \Rightarrow R) \equiv (P \vee Q) \Rightarrow R$
(h) $(P \Rightarrow R) \vee (Q \Rightarrow R) \equiv (P \wedge Q) \Rightarrow R$
Solution:

P	Q	R	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \Rightarrow R) \vee (Q \Rightarrow R)$
T	T	T	T	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

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P	Q	R	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

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7. Use truth tables to show the following logical equivalences for biconditional statements.

   (a) $P \Leftrightarrow Q \equiv Q \Leftrightarrow P$

   (b) $P \Leftrightarrow Q \equiv \sim Q \Leftrightarrow \sim P$

   Solution:

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

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P	Q	$\sim Q$	$\sim P$	$\sim Q \Leftrightarrow \sim P$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	F
F	F	T	T	T

   (c) $P \Leftrightarrow (Q \Leftrightarrow R) \equiv (P \Leftrightarrow Q) \Leftrightarrow R$

   Solution:

P	Q	R	$Q \Leftrightarrow R$	$P \Leftrightarrow (Q \Leftrightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	T	T
F	T	T	T	F
F	T	F	F	T
F	F	T	F	T
F	F	F	T	F

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P	Q	R	$P \Leftrightarrow Q$	$(P \Leftrightarrow Q) \Leftrightarrow R$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	F	T
F	T	T	F	F
F	T	F	F	T
F	F	T	T	T
F	F	F	T	F

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   (d) $(P \Rightarrow Q) \wedge (Q \Rightarrow P) \equiv (P \Leftrightarrow Q)$

   (e) $P \Leftrightarrow Q \equiv \sim (P \wedge Q) \vee (\sim P \wedge \sim Q)$

   (f) $\sim (P \Leftrightarrow Q) \equiv P \Leftrightarrow \sim Q$

8. Recall that a tautology is a statement that is true for every possible assignment of truth values, and a contradiction is a statement that is false for every possible assignment. Figure out which of the following statements are tautologies, contradictions, or neither. You can use a truth table if you like, but you may not need to.

   (a) $P \Rightarrow P$

   (b) $P \Rightarrow \sim P$ [Answer: Neither]

   (c) $P \Rightarrow (P \vee Q)$

   (d) $(P \wedge Q) \Rightarrow Q$ [Answer: Tautology. We could make a truth table, or use the fact that if both P and Q are true then Q must be true.]

   (e) $(P \Rightarrow Q) \vee (Q \Rightarrow P)$

   (f) $P \Leftrightarrow \sim P$ [Answer: Contradiction]

   (g) $P \Rightarrow (P \Leftrightarrow \sim P)$

   (h) $P \Rightarrow (P \Leftrightarrow Q)$ [Answer: Neither. The statement may be true or false. It's certainly true if both P and Q are true. If P is true and Q is false, however, the statement would be false.]

   (i) $(P \Leftrightarrow Q) \Rightarrow (P \Rightarrow Q)$ [Answer: Tautology. Suppose $P \Leftrightarrow Q$ is true so P and Q have the same truth values. Then it is impossible for the second half to be false because that would require P to be true and Q to be false.]

(j) $((P \Rightarrow Q) \wedge (Q \Rightarrow P)) \Rightarrow (P \Leftrightarrow Q)$

9. Assume each of the sentences below are false. What can you conclude about the truth values of the parts? Note that there are two ways to do this. You can either draw a truth tables and see where the statements are false or you can try to reason through from what you know about condition statements.

   (a) $(P \wedge Q) \Rightarrow R$

   (b) $P \Rightarrow (Q \vee R)$

   (c) $P \vee (P \Leftrightarrow Q)$

   (d) $\sim P \Rightarrow (P \Leftrightarrow \sim R)$ [Answer: P and R must both be false.]

   (e) $\sim (\sim P \vee Q) \Rightarrow (\sim R)$

   (f) $(\sim P \Leftrightarrow Q) \wedge \sim Q$ [Answer: P must be true and Q must be false.]

10. When the parenthesis are left out we can rely on an "order of operations" just as we do for addition, multiplication and the other standard operations on real numbers. We use $\sim \; > \; \wedge \; > \; \vee \; > \Rightarrow \; > \Leftrightarrow$ when there are no parenthesis. This means $\sim$ takes priority over all the others, with $\wedge$ second and so on. When operations are equal we work from left to right. Place parenthesis according to this order of operations to remove any possible ambiguity for an outsider without this knowledge.

    (a) $P \wedge Q \vee R$ [Answer: $(P \wedge Q) \vee R$.]

    (b) $P \vee Q \wedge R$ [Answer: $P \vee (Q \wedge R)$.]

    (c) $P \wedge \sim Q \vee \sim R$ [Answer: $(P \wedge (\sim Q)) \vee (\sim R)$.]

    (d) $\sim P \vee \sim Q \vee \sim P$ [Answer: $((\sim P) \wedge (\sim Q)) \vee (\sim P)$.]

    (e) $P \Rightarrow Q \wedge P \Rightarrow R$ [Answer: $(P \Rightarrow (Q \wedge P)) \Rightarrow R$.]

    (f) $P \Leftrightarrow Q \wedge P \Rightarrow R$ [Answer: $P \Leftrightarrow ((Q \wedge P) \Rightarrow R)$.]

1.3 Quantifiers

Since mathematical statements can be communicated through both language and symbols in many ways, note that there will be many correct answers to some of the following questions.

1. Translate the following English sentences into statements involving open sentences with quantifiers. All these statements are true over the universe of positive whole numbers.
   (a) There is a number less than two.
   (b) Every number is bigger than or equal to one. [Answer: $(\forall n)(n > 1) \vee (n = 1)$. We could also write $(\forall n)(n \geq 1)$.]
   (c) There is a number bigger than two and less than eight.
   (d) Every number has a number bigger than it. [Answer: $(\forall n)(\exists m)(m > n)$.]
   (e) Every number has a square. [Answer: $(\forall n)(\exists m)(m = n^2)$.]
   (f) The sum of two numbers is bigger than either of the numbers. [Answer: $(\forall m)(\forall n)(m + n > m) \wedge (m + n > n)$.]
2. Translate the following statements into simple English sentences. All these statements are true over the universe of positive whole numbers.
   (a) $(\forall x)(x^2 \geq x)$ [Answer: The square of number is greater than or equal to that number.]
   (b) $(\forall x)(x < 5) \vee (x > 4)$.
   (c) $(\forall x)(\exists y)(y \leq x)$
   (d) $(\forall x)(\forall y)(\exists z)(x + y = z)$ [Answer: The sum of two positive whole numbers is a positive whole number.]
   (e) $(\forall x)(\forall y)(\exists z)(xy = z)$
   (f) $(\exists x)(\forall y)(xy = y)$
   (g) $(\exists x)(x \text{ is prime}) \wedge (x > 100)$ [Answer: There is a prime number bigger than a hundred.]
   (h) $(\forall x)(\exists y)(y \text{ is prime }) \wedge (y > x)$ [Answer: Every number is less than some prime number.]
3. The following statements are false over the universe of positive whole numbers. Negate them, and pass the negation through the quantifiers to construct a new and true statement. Make sure you understand why this new statement is true.
   (a) $(\forall x)(x < 5)$
   (b) $(\exists x)(x < 5) \wedge (x > 5)$.
   (c) $(\forall x)(\exists y)(y < x)$
   (d) $(\forall x)(\forall y)(xy = y)$ [Answer: $(\exists x)(\exists y)(xy \neq y)$. Here, any x except for one does the trick.]
   (e) $(\exists x)(x \text{ is prime}) \wedge (x < 2)$ [Answer: $(\forall x)(x \text{ fails to be prime}) \vee (x \geq 2.)$]
   (f) There is a prime number between 200 and 210. [Answer: $(\forall n)(n \leq 200) \vee (n \geq 210) \vee \sim (n \text{ prime})$.]
   (g) $(\forall x)(\forall y)(\forall z)(x + y = z)$

(h) $(\forall x)(\forall z)(\exists y)(x + y = z)$

(i) $(\forall z)(\exists x)(\exists y)(x + y = z)$ [Answer: $(\exists z)(\forall x)(\forall y)(x + y \neq z)$. This is definitely true as z is allowed to equal one.]

4. Translate the following statements involving open sentences with quantifiers into normal English sentences then state whether the sentence is true or false for the collection of all whole numbers (both positive and negative.)

   (a) $(\forall x)(2x \neq x)$ [Answer: No number is equal to its double. False. (Remember zero.)]

   (b) $(\forall x)(x \text{ is even}) \Rightarrow (x^2 \text{ is even})$

   (c) $(\forall x)(x^2 \text{ is even}) \Rightarrow (x \text{ is even})$ [Answer: If the square of a number is even then that number must be even. True.]

   (d) $(\forall x)(x \text{ is odd}) \Rightarrow (x^3 \text{ is odd})$

   (e) $(\forall x)(x \text{ is prime}) \wedge (x \text{ is even}) \Rightarrow (x = 2)$ [Answer: An even prime must equal 2. True]

   (f) $(\exists x)(\exists y)\ y < x$

   (g) $(\forall x)(\exists y)\ y < x$

   (h) $(\exists x)(\forall y)\ y < x$

   (i) $(\forall x)(\exists y)\ x < y < 2x$ [Answer: Given any number, we can find another number between that number and its double. False. (This is only true for numbers bigger than one.)]

   (j) $(\forall x)(\forall y)(x + y > x) \wedge (x + y > y)$ [Answer: The sum of two numbers is always bigger than either of the numbers. False. (Remember the negatives.)]

   (k) $(\forall x)(\forall y)(\exists z)\ y \neq x \Rightarrow (x < z < y) \vee (y < z < x)$ [Answer: For any two different integers, there is an integer between them. False.]

5. State whether the following sentences are true over the different universes $U_1 = \mathbb{N}$, $U_2 = \mathbb{Z}$, and $U_3 =$ the set of prime numbers.

   (a) $(\exists x)\ x < 2$

   (b) $(\forall x)\ x \geq 2$

   (c) $(\exists x)\ x > 89 \wedge x < 97$

   (d) $(\forall x)(\exists y)\ y < x$ [Answers: $U_1 = F, U_2 = T, U_3 = F$]

   (e) $(\forall x)(\exists y)\ y > x$ [Answers: $U_1 = T, U_2 = T, U_3 = T$]

   (f) $(\exists x)(\forall y)\ y < x$

   (g) $(\exists x)(\forall y)\ y > x$

   (h) $(\exists x)(\forall y)\ y \geq x$

   (i) $(\forall x)(\forall y)(\exists z)\ x + y = z$ [Answers: $U_1 = T, U_2 = T, U_3 = F$]

   (j) $(\forall x)(\forall y)(\exists z)\ x - y = z$

   (k) $(\forall x)(\forall y)\ y \neq 2x$ [Answers: $U_1 = F, U_2 = F, U_3 = T$]

6. State whether the following sentences are true over the real numbers.

(a) $(\exists x)(\exists y)\ y < x$ [Answer: True]

(b) $(\exists x)(\forall y)\ y < x$ [Answer: False]

(c) $(\forall x)(\exists y)\ y < x$ [Answer: True]

(d) $(\forall x)(\forall y)\ y < x$ [Answer: False]

(e) $(\exists x)(\exists y)\ y = x^2$ [Answer: True]

(f) $(\exists x)(\forall y)\ y = x^2$ [Answer: False]

(g) $(\forall x)(\exists y)\ y = x^2$ [Answer: True]

(h) $(\forall x)(\forall y)\ y = x^2$ [Answer: False]

(i) $(\forall y)(\exists x)\ y = x^2$ [Answer: False]

(j) $(\forall x)(\exists y)\ 0 < y < x$ [Answer: False]

(k) $(\forall x)(\forall y)\ xy = 0$ [Answer: False]

(l) $(\exists x)(\forall y)\ xy = 0$ [Answer: True]

(m) $(\forall x)(\exists y)\ xy = 0$ [Answer: True]

(n) $(\exists x)(\exists y)\ xy = 0$ [Answer: True]

(o) $(\forall x)(\exists y)\ xy = 1$ [Answer: False]

(p) $(\exists x)(\exists y)\ xy = 1$ [Answer: True]

(q) $(\exists x)(\forall y)\ xy = 1$ [Answer: False]

(r) $(\exists x)\ x^2 > x$ [Answer: True]

(s) $(\exists x)\ x/2 < x$ [Answer: True]

(t) $(\forall x)\ x^2 > x$ [Answer: False]

(u) $(\forall x)\ x/2 < x$ [Answer: False]

(v) $(\forall x)(\exists y)\ xy < 1$[Answer: True]

(w) $(\exists x)(\exists y)\ xy < 1$[Answer: True]

(x) $(\exists x)(\forall y)\ xy < 1$[Answer: True]

(y) $(\forall n)(\exists x)(\exists y)\ (x+y)^n = x^n + y^n$ [Answer: True]

(z) $(\exists n)(\forall x)(\forall y)\ (x+y)^n = x^n + y^n$ [Answer: True]

1.4 Sets, Subsets, and Cardinality

1. Find the cardinality of the following sets. Remember this is the same as asking for the number of elements in the set.

   (a) $\{1, 2\}$ [Answer: 2]
   (b) $\{2, 3\}$ [Answer: 2]
   (c) $\{1, \{1, 2\}\}$ [Answer: 2]
   (d) $\{\{1\}, \{1, 2\}\}$ [Answer: 2]
   (e) $\{\{1, 2\}\}$ [Answer: 1]
   (f) $\{\{1, 2, 3, 4\}\}$ [Answer: 1]
   (g) $\{1, \{1\}, \{\{1\}\}, \{\{\{1\}\}\}\}$ [Answer: 4]
   (h) $\{\{1, \{1\}, \{\{1\}\}, \{\{\{1\}\}\}\}\}$ [Answer: 1]
   (i) $\{1, 2, \{1\}, \{2\}, \{1, 2\}\}$ [Answer: 5]
   (j) $\{1, 2, \{1\}, \{2\}, \{1, 2\}, \{\{1\}, 2\}, \{1, \{2\}\}, \{\{1, 2\}\}\}$ [Answer: 8]
   (k) $\emptyset$ [Answer: 0]
   (l) $\{\emptyset\}$ [Answer: 1]
   (m) $\{\{\emptyset\}\}$ [Answer: 1]
   (n) $\{\{\{\emptyset\}\}\}$ [Answer: 1]
   (o) $\{\{\emptyset, \{\emptyset\}\}\}$ [Answer: 1]
   (p) $\{\emptyset, \{\emptyset\}\}$ [Answer: 2]
   (q) $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ [Answer: 3]
   (r) $\{\{\emptyset, \{\emptyset\}\}, \{\{\emptyset\}\}\}$ [Answer: 2]
   (s) $\{\emptyset, \{\{\emptyset\}, \{\{\emptyset\}\}\}\}$ [Answer: 2]
   (t) $\{\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}\}$ [Answer: 1]

2. State whether the following are true or false:

   (a) $1 \in \{1\}$ [Answer: True]
   (b) $1 \in \{1, 2\}$ [Answer: True]
   (c) $1 \in \{\{1\}\}$ [Answer: False]
   (d) $1 \in \{\{1\}, 1\}$ [Answer: True]
   (e) $1 \in \{\{1\}, 2\}$ [Answer: False]
   (f) $1 \in \{\{2\}, 1\}$ [Answer: True]
   (g) $\{1\} \in \{1\}$ [Answer: False]
   (h) $\{1\} \in \{\{1\}\}$ [Answer: True]
   (i) $\{1\} \in \{\{\{1\}\}\}$ [Answer: False]

(j) $\{\{1\}\} \in \{\{\{1\}\}\}$ [Answer: True]
(k) $\{1,2\} \in \{\{1\},\{2\}\}$ [Answer: False]
(l) $\{1,2\} \in \{\{1,2\}\}$ [Answer: True]
(m) $\{1,2\} \in \{\{1\},\{2\},\{1,2\}\}$ [Answer: True]
(n) $\{1,2\} \in \{\{\{1\},\{2\}\}\}$ [Answer: False]
(o) $\{1,2\} \in \{\{\{1\},\{2\},\{1,2\}\}\}$ [Answer: False]
(p) $\{1,2\} \in \{\{\{1\},\{2\}\},\{1,2\}\}$ [Answer: True]
(q) $\{1,2\} \in \{\{\{1\},\{2\},\{1,2\}\},\{1,2\}\}$ [Answer: True]
(r) $\{1,2\} \in \{\{\{1\},\{2\},\{1,2\}\},\{1\},\{2\},\{1,2\}\}$ [Answer: True]
(s) $\{\{1,2\}\} \in \{\{\{1\},\{2\},\{1,2\}\},\{1\},\{2\},\{1,2\}\}$ [Answer: False]
(t) $\{\{1\},\{2\},\{1,2\}\} \in \{\{\{1\},\{2\},\{1,2\}\},\{1\},\{2\},\{1,2\}\}$ [Answer: True]
(u) $1 \in \{\{\{1\},\{2\},\{1,2\}\},\{1\},\{2\},\{1,2\}\}$ [Answer: False]

3. State whether the following are true or false:

(a) $\emptyset \in \emptyset$ [Answer: False]
(b) $\emptyset \in \{\emptyset\}$ [Answer: True]
(c) $\emptyset \in \{\{\emptyset\}\}$ [Answer: False]
(d) $\emptyset \in \{\{\{\emptyset\}\}\}$ [Answer: False]
(e) $\{\emptyset\} \in \emptyset$ [Answer: False]
(f) $\{\emptyset\} \in \{\emptyset\}$ [Answer: False]
(g) $\{\emptyset\} \in \{\{\emptyset\}\}$ [Answer: True]
(h) $\{\emptyset\} \in \{\{\{\emptyset\}\}\}$ [Answer: False]
(i) $\emptyset \in \{\emptyset,\{\emptyset\}\}$ [Answer: True]
(j) $\{\emptyset\} \in \{\emptyset,\{\emptyset\}\}$ [Answer: True]
(k) $\emptyset \in \{\{\emptyset,\{\emptyset\}\}\}$ [Answer: False]
(l) $\{\emptyset\} \in \{\{\emptyset,\{\emptyset\}\}\}$ [Answer: False]
(m) $\{\emptyset,\{\emptyset\}\} \in \{\{\emptyset,\{\emptyset\}\}\}$ [Answer: True]
(n) $\{\emptyset,\{\emptyset\}\} \in \{\emptyset,\{\emptyset,\{\emptyset\}\}\}$ [Answer: True]
(o) $\emptyset \in \{\emptyset,\{\emptyset,\{\emptyset\}\}\}$ [Answer: True]
(p) $\{\emptyset\} \in \{\emptyset,\{\emptyset,\{\emptyset\}\}\}$ [Answer: False]
(q) $\{\emptyset\} \in \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}$ [Answer: True]
(r) $\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\} \in \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}$ [Answer: False]

4. State whether the following are true or false:

(a) $1 \subseteq \{1\}$ [Answer: False]
(b) $1 \subseteq \{\{1\}\}$ [Answer: False]

(c) $\{1\} \subseteq \{1\}$ [Answer: True]
(d) $\{1\} \subseteq \{\{1\}\}$ [Answer: False]
(e) $\{1\} \subseteq \{1, \{1\}\}$ [Answer: True]
(f) $\{1\} \subseteq \{1, 2, \{1\}, \{2\}, \{1, 2\}\}$ [Answer: True]
(g) $\{1, 2\} \subseteq \{1, 2, \{1\}, \{2\}, \{1, 2\}\}$ [Answer: True]
(h) $\{1, 2\} \subseteq \{2, \{1\}, \{2\}, \{1, 2\}\}$ [Answer: False]
(i) $\{1, 2\} \subseteq \{1, \{1\}, \{2\}, \{1, 2\}\}$ [Answer: False]
(j) $\{1\} \subseteq \{\{\{1\}\}\}$ [Answer: False]
(k) $\{\{1\}\} \subseteq \{1, \{\{1\}\}\}$ [Answer: False]
(l) $\{\{1\}\} \subseteq \{1, \{1\}, \{\{1\}\}\}$ [Answer: True]
(m) $\{1, 2\} \subseteq \{1, 2\}$ [Answer: True]
(n) $\{1, 2\} \subseteq \{\{1\}, \{2\}\}$ [Answer: False]
(o) $\{1, 2\} \subseteq \{\{1, 2\}\}$ [Answer: False]
(p) $\{1, 2\} \subseteq \{1, 2, \{1\}, \{2\}, \{1, 2\}\}$ [Answer: True]
(q) $\{1, 2\} \subseteq \{\{\{1\}, \{2\}, \{1, 2\}\}\}$ [Answer: False]
(r) $\{1, 2\} \subseteq \{\{\{1\}, \{2\}, \{1, 2\}\}, \{1, 2\}\}$ [Answer: False]
(s) $\{1, 2\} \subseteq \{\{\{1\}, \{2\}, \{1, 2\}\}, \{1\}, \{2\}, \{1, 2\}\}$ [Answer: False]
(t) $\{\{1, 2\}\} \subseteq \{\{\{1\}, \{2\}\}, \{1, 2\}\}$ [Answer: True]
(u) $\{\{1, 2\}\} \subseteq \{1, 2, \{1\}, \{2\}\}$ [Answer: False]
(v) $\{\{1, 2\}\} \subseteq \{\{\{1\}, \{2\}, \{1, 2\}\}, \{1\}, \{2\}, \{1, 2\}\}$ [Answer: True]
(w) $\{1, \{2\}\} \subseteq \{1, 2, \{1\}, \{2\}, \{1, 2\}\}$ [Answer: True]
(x) $\{1, \{2\}\} \subseteq \{\{\{1\}, \{2\}, \{1, 2\}\}, \{1\}, \{2\}, \{1, 2\}\}$ [Answer: False]
(y) $\{1, \{2\}\} \subseteq \{1, 2, \{1, 2\}\}$ [Answer: False]
(z) $\{\{1\}, \{2\}, \{1, 2\}\} \subseteq \{\{\{1\}, \{2\}, \{1, 2\}\}, \{1\}, \{2\}, \{1, 2\}\}$ [Answer: True]

5. State whether the following are true or false:

(a) $\emptyset \subseteq \emptyset$ [Answer: True]
(b) $\emptyset \subseteq \{\emptyset\}$ [Answer: True]
(c) $\emptyset \subseteq \{\{\emptyset\}\}$ [Answer: True]
(d) $\emptyset \subseteq \{\{\{\emptyset\}\}\}$ [Answer: True]
(e) $\{\emptyset\} \subseteq \emptyset$ [Answer: False]
(f) $\{\emptyset\} \subseteq \{\emptyset\}$ [Answer: True]
(g) $\{\emptyset\} \subseteq \{\{\emptyset\}\}$ [Answer: False]
(h) $\{\emptyset\} \subseteq \{\{\{\emptyset\}\}\}$ [Answer: False]
(i) $\emptyset \subseteq \{\emptyset, \{\emptyset\}\}$ [Answer: True]

(j) $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$ [Answer: True]

(k) $\emptyset \subseteq \{\{\emptyset, \{\emptyset\}\}\}$ [Answer: True]

(l) $\{\emptyset\} \subseteq \{\{\emptyset, \{\emptyset\}\}\}$ [Answer: False]

(m) $\{\emptyset, \{\emptyset\}\} \subseteq \{\{\emptyset, \{\emptyset\}\}\}$ [Answer: False]

(n) $\{\emptyset, \{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ [Answer: False]

(o) $\emptyset \subseteq \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ [Answer: True]

(p) $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$ [Answer: True]

(q) $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ [Answer: True]

(r) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \subseteq \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ [Answer: True]

(s) $\{\emptyset, \{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ [Answer: True]

(t) $\{\emptyset, \{\emptyset, \{\emptyset\}\}\} \subseteq \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ [Answer: True]

(u) $\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \subseteq \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ [Answer: True]

(v) $\{\emptyset, \{\emptyset\}\} \subseteq \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ [Answer: False]

(w) $\{\emptyset, \{\emptyset, \{\emptyset\}\}\} \subseteq \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ [Answer: False]

(x) $\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \subseteq \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ [Answer: True]

(y) $\{\{\emptyset, \{\emptyset\}\}\} \subseteq \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ [Answer: True]

(z) $\{\{\emptyset\}\} \subseteq \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ [Answer: True]

1.5 Power Sets

1. Find the cardinality of the following sets. Recall that the power set of a set with n elements always has 2^n elements.

   (a) $\mathcal{P}(\{1\})$ [Answer: 2]
   (b) $\mathcal{P}(\{1,2\})$ [Answer: 4]
   (c) $\mathcal{P}(\{1,2,3,4\})$ [Answer: 16]
   (d) $\mathcal{P}(\{1,\{1\}\})$ [Answer: 4]
   (e) $\mathcal{P}(\emptyset)$ [Answer: 1]
   (f) $\mathcal{P}(\{\emptyset\})$ [Answer: 2]
   (g) $\mathcal{P}(\{\{\emptyset\}\})$ [Answer: 2]
   (h) $\mathcal{P}(\{\{\{\emptyset\}\}\})$ [Answer: 2]
   (i) $\mathcal{P}(\{\{\emptyset,\{\emptyset\}\}\})$ [Answer: 2]
   (j) $\mathcal{P}(\{\emptyset,\{\emptyset\}\})$ [Answer: 4]
   (k) $\mathcal{P}(\{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\})$ [Answer: 8]
   (l) $\mathcal{P}(\{\{\emptyset,\{\emptyset\}\},\{\{\emptyset\}\}\})$ [Answer: 4]
   (m) $\mathcal{P}(\{\emptyset,\{\{\emptyset\},\{\{\emptyset\}\}\}\})$ [Answer: 4]
   (n) $\mathcal{P}(\{\{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\}\})$ [Answer: 2]

2. State whether the following statements are true or false.

   (a) $1 \in \mathcal{P}(\{1,2\})$ [Answer: False]
   (b) $\{1\} \in \mathcal{P}(\{1\})$ [Answer: True]
   (c) $\{1\} \in \mathcal{P}(\{1,2\})$ [Answer: True]
   (d) $\{1\} \in \mathcal{P}(\{\{1,2\}\})$ [Answer: False]
   (e) $\{1,2\} \in \mathcal{P}(\{\{1,2\}\})$ [Answer: False]
   (f) $\{1,2\} \in \mathcal{P}(\{\{1\},\{2\}\})$ [Answer: False]
   (g) $\{1,2\} \in \mathcal{P}(\{1,2\})$ [Answer: True]
   (h) $\{\{1\}\} \in \mathcal{P}(\{\{1,2\}\})$ [Answer: False]
   (i) $\{\{1\}\} \in \mathcal{P}(\{\{1\},\{2\}\})$ [Answer: True]
   (j) $\{1,2\} \in \mathcal{P}(\{1,2,\{1,2\}\})$ [Answer: True]

3. State whether the following statements are true or false.

   (a) $\emptyset \in \mathcal{P}(\emptyset)$ [Answer: True]
   (b) $\emptyset \in \mathcal{P}(\{\{\{\emptyset\}\}\})$ [Answer: True]
   (c) $\{\emptyset\} \in \mathcal{P}(\emptyset)$ [Answer: False]
   (d) $\{\emptyset\} \in \mathcal{P}(\{\emptyset\})$ [Answer: True]

(e) $\{\emptyset\} \in \mathcal{P}(\{\{\emptyset\}\})$ [Answer: False]
(f) $\{\emptyset\} \in \mathcal{P}(\{\{\emptyset, \{\emptyset\}\}\})$ [Answer: False]
(g) $\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(\{\{\emptyset, \{\emptyset\}\}\})$ [Answer: False]
(h) $\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(\{\emptyset, \{\emptyset, \{\emptyset\}\}\})$ [Answer: False]
(i) $\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\})$ [Answer: True]
(j) $\{\emptyset, \{\emptyset, \{\emptyset\}\}\} \in \mathcal{P}(\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\})$ [Answer: True]

4. State whether the following statements are true or false.

(a) $\{1\} \subseteq \mathcal{P}(\{1\})$ [Answer: False]
(b) $\{1\} \subseteq \mathcal{P}(\{\{1\}\})$ [Answer: False]
(c) $\{\{1\}\} \subseteq \mathcal{P}(\{1\})$ [Answer: True]
(d) $\{\{1\}\} \subseteq \mathcal{P}(\{1, 2\})$ [Answer: True]
(e) $\{\{1, 2\}\} \subseteq \mathcal{P}(\{1, 2\})$ [Answer: True]
(f) $\{\{1, 2\}\} \subseteq \mathcal{P}(\{\{1, 2\}\})$ [Answer: False]
(g) $\{\{1\}, \{2\}\} \subseteq \mathcal{P}(\{1, 2\})$ [Answer: True]
(h) $\{\{1\}, \{2\}\} \subseteq \mathcal{P}(\{\{1, 2\}\})$ [Answer: False]
(i) $\{\{1\}, \{2\}, \{1, 2\}\} \subseteq \mathcal{P}(\{1, 2\})$ [Answer: True]
(j) $\{\{1\}, \{2\}, \{1, 2\}\} \subseteq \mathcal{P}(\{\{1, 2, \{1\}\}\})$ [Answer: True]
(k) $\{\{1\}, \{2\}, \{1, 2\}\} \subseteq \mathcal{P}(\{1, 2, \{1\}\})$ [Answer: True]

5. State whether the following statements are true or false.

(a) $\emptyset \subseteq \mathcal{P}(\emptyset)$ [Answer: True]
(b) $\{\emptyset\} \subseteq \mathcal{P}(\emptyset)$ [Answer: True]
(c) $\{\emptyset\} \subseteq \mathcal{P}(\{\emptyset\})$ [Answer: True]
(d) $\{\emptyset\} \subseteq \mathcal{P}(\{\{\emptyset\}\})$ [Answer: True]
(e) $\{\{\emptyset\}\} \subseteq \mathcal{P}(\{\emptyset\})$ [Answer: True]
(f) $\{\emptyset, \{\emptyset\}\} \subseteq \mathcal{P}(\{\emptyset\})$ [Answer: True]
(g) $\{\emptyset, \{\{\emptyset\}\}\} \subseteq \mathcal{P}(\{\emptyset\})$ [Answer: False]
(h) $\{\{\emptyset, \{\emptyset\}\}\} \subseteq \mathcal{P}(\{\emptyset\})$ [Answer: False]
(i) $\mathcal{P}(\emptyset) \subseteq \mathcal{P}(\emptyset)$ [Answer: True]
(j) $\mathcal{P}(\emptyset) \subseteq \mathcal{P}(\{\emptyset\})$ [Answer: True]
(k) $\mathcal{P}(\emptyset) \subseteq \mathcal{P}(\{\{\emptyset\}\})$ [Answer: True]
(l) $\mathcal{P}(\{\emptyset\}) \subseteq \mathcal{P}(\emptyset)$ [Answer: False]
(m) $\mathcal{P}(\{\emptyset\}) \subseteq \mathcal{P}(\{\emptyset\})$ [Answer: True]
(n) $\mathcal{P}(\{\emptyset\}) \subseteq \mathcal{P}(\{\{\emptyset\}\})$ [Answer: False]
(o) $\mathcal{P}(\{\{\emptyset\}\}) \subseteq \mathcal{P}(\{\emptyset\})$ [Answer: False]

(p) $\mathcal{P}(\{\emptyset, \{\emptyset\}\}) \subseteq \mathcal{P}(\{\emptyset\})$ [Answer: False]

(q) $\mathcal{P}(\{\emptyset\}) \subseteq \mathcal{P}(\{\emptyset, \{\emptyset\}\})$ [Answer: True]

(r) $\{\{\emptyset, \{\emptyset\}\}\} \subseteq \{\mathcal{P}(\{\emptyset\})\}$ [Answer: True]

6. Find the cardinality of the following sets.

(a) $\mathcal{P}(\mathcal{P}(\{1\})$ [Answer: 4]

(b) $\mathcal{P}(\mathcal{P}(\{\{1\}\})$ [Answer: 4]

(c) $\mathcal{P}(\mathcal{P}(\{\{1,2\}\})$ [Answer: 4]

(d) $\mathcal{P}(\mathcal{P}(\{1,2\})$ [Answer: 16]

(e) $\mathcal{P}(\mathcal{P}(\emptyset))$ [Answer: 2]

(f) $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$ [Answer: 4]

(g) $\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))))$ [Answer: 16]

(h) $\mathcal{P}(\mathcal{P}(\{\emptyset\}))$ [Answer: 4]

(i) $\mathcal{P}(\mathcal{P}(\mathcal{P}(\{\emptyset\})))$ [Answer: 16]

(j) $\mathcal{P}(\mathcal{P}(\mathcal{P}(\{\emptyset, \{\emptyset\}\})))$ [Answer: 65536]

1.6 Cartesian Products

1. Let $A = \{1, 2, 4\}$ be the positive divisors of four, $B = \{1, 5\}$ be the positive divisors of five, and $C = \{1\}$ be the positive divisors of one. Find the following sets.

   (a) $A \times B$ [Answer: $\{(1, 1), (1, 5), (2, 1), (2, 5), (4, 1), (4, 5)\}$.]

   (b) $A \times C$ [Answer: $\{(1, 1), (2, 1), (4, 1)\}$.]

   (c) $B \times C$ [Answer: $\{(1, 1), (5, 1)\}$.]

   (d) $B \times B$ [Answer: $\{(1, 1), (5, 1), (1, 5), (5, 5)\}$.]

   (e) $C \times C$ [Answer: $\{(1, 1)\}$.]

   (f) $(A \cup B) \times C$ [Answer: $\{1, 2, 4, 5\} \times C = \{(1, 1), (2, 1), (4, 1), (5, 1)\}$.]

   (g) $(A \times C) \cup (B \times C)$ [Answer: $\{(1, 1), (2, 1), (4, 1)\} \cup \{(1, 1), (5, 1)\} = \{(1, 1), (2, 1), (4, 1), (5, 1)\}$.]

   (h) $(A \cap B) \times C$ [Answer: $\{1\} \times C = \{(1, 1)\}$.]

   (i) $(A \times C) \cap (B \times C)$ [Answer: $\{(1, 1), (2, 1), (4, 1)\} \cap \{(1, 1), (5, 1)\} = \{(1, 1)\}$.]

   (j) $A \times (B - C)$ [Answer: $A \times \{5\} = \{(1, 5), (2, 5), (4, 5)\}$.]

   (k) $(A\times B)-(A\times C)$ [Answer: $\{(1, 1), (1, 5), (2, 1), (2, 5), (4, 1), (4, 5)\}-\{(1, 1), (2, 1), (4, 1)\} = \{(1, 5), (2, 5), (4, 5)\}$.]

   (l) $A \times \emptyset$ [Answer: $\emptyset$.]

   (m) $\{A \times \{\emptyset\}\}$ [Answer: $\{(1, \emptyset), (2, \emptyset), (4, \emptyset)\}$.]

   (n) $\{A \times \{\{\emptyset\}\}\}$ [Answer: $\{(1, \{\emptyset\}), (2, \{\emptyset\}), (4, \{\emptyset\})\}$.]

   (o) $\{A \times \{\emptyset, \{\emptyset\}\}\}$ [Answer: $\{(1, \emptyset), (2, \emptyset), (4, \emptyset), (1, \{\emptyset\}), (2, \{\emptyset\}), (4, \{\emptyset\})\}$.]

   (p) $\emptyset \times \emptyset$ [Answer: $\emptyset$.]

   (q) $\{\emptyset\} \times \emptyset$ [Answer: $\emptyset$.]

   (r) $\emptyset \times \{\emptyset\}$ [Answer: $\emptyset$.]

   (s) $\{\emptyset\} \times \{\emptyset\}$ [Answer: $\{(\emptyset, \emptyset)\}$.]

   (t) $\{\emptyset, \{\emptyset\}\} \times \{\emptyset\}$ [Answer: $\{(\emptyset, \emptyset), (\{\emptyset\}, \emptyset)\}$.]

   (u) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \times \emptyset$ [Answer: $\emptyset$.]

   (v) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \times \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ [Answer: $\{(\emptyset, \emptyset), (\emptyset, \{\emptyset\}), (\emptyset, \{\emptyset, \{\emptyset\}\}), (\{\emptyset\}, \emptyset), (\{\emptyset\}, \{\emptyset\}), (\{\emptyset\}, \{\emptyset, \{\emptyset\}\}), (\{\emptyset, \{\emptyset\}\}, \emptyset), (\{\emptyset, \{\emptyset\}\}, \{\emptyset\}), (\{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\})\}$.]

2. Find the cardinality of the following sets:

   (a) $\{1\} \times \{1\}$ [Answer: 1]

   (b) $\{1\} \times \{2\}$ [Answer: 1]

   (c) $\{1\} \times \{1, 2\}$ [Answer: 2]

   (d) $\{1, 2\} \times \{1, 2\}$ [Answer: 4]

   (e) $\{1, 2\} \times \{\{1\}, \{2\}\}$ [Answer: 4]

   (f) $\{1, 2\} \times \{\{1, 2\}\}$ [Answer: 2]

   (g) $\{\{1, 2\}\} \times \{\{1, 2\}\}$ [Answer: 1]

(h) $\{\{1\},2\} \times \{1,\{2\}\}$ [Answer: 4]
(i) $\{1,2,3\} \times \{1,2\}$ [Answer: 6]
(j) $\{\{1\},\{2\},\{1,2\}\} \times \{1,\{2\}\}$ [Answer: 6]
(k) $\emptyset \times \emptyset$ [Answer: 0]
(l) $\emptyset \times \{\emptyset\}$ [Answer: 0]
(m) $\{\emptyset\} \times \emptyset$ [Answer: 0]
(n) $\emptyset \times \{\{\emptyset\}\}$ [Answer: 0]
(o) $\emptyset \times \{\emptyset,\{\emptyset\}\}$ [Answer: 0]
(p) $\{\emptyset\} \times \{\emptyset\}$ [Answer: 1]
(q) $\{\emptyset\} \times \{\{\emptyset\}\}$ [Answer: 1]
(r) $\{\{\emptyset\}\} \times \{\{\emptyset\}\}$ [Answer: 1]
(s) $\{\emptyset\} \times \{\emptyset,\{\emptyset\}\}$ [Answer: 2]
(t) $\{\emptyset,\{\emptyset\}\} \times \{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\}$ [Answer: 6]
(u) $\{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\} \times \{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\}$ [Answer: 9]
(v) $\{\{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\}\} \times \{\{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\}\}$ [Answer: 1]
(w) $\{\{\{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\}\}\} \times \{\{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\}\}\}$ [Answer: 1]

3. Find the cardinality of the following sets:

(a) $\mathcal{P}(\{1\} \times \{2\})$ [Answer: 2]
(b) $\mathcal{P}(\{1\} \times \{1,2\})$ [Answer: 4]
(c) $\mathcal{P}(\{1\} \times \{1,2,3\})$ [Answer: 8]
(d) $\mathcal{P}(\{1,2,3\} \times \{1,2,3\})$ [Answer: 512]
(e) $\mathcal{P}(\{1,2,3\} \times \{\{1,2,3\}\})$ [Answer: 8]
(f) $\mathcal{P}(\emptyset \times \emptyset)$ [Answer: 1]
(g) $\mathcal{P}(\emptyset \times \{\{\emptyset\}\})$ [Answer: 1]
(h) $\mathcal{P}(\{\emptyset\} \times \{\emptyset\})$ [Answer: 2]
(i) $\mathcal{P}(\{\{\emptyset\}\} \times \{\{\emptyset\}\})$ [Answer: 2]
(j) $\mathcal{P}(\{\emptyset\} \times \{\emptyset,\{\emptyset\}\})$ [Answer: 4]
(k) $\mathcal{P}(\{\emptyset,\{\emptyset\}\} \times \{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\})$ [Answer: 64]
(l) $\mathcal{P}(\{1\}) \times \mathcal{P}(\{2\})$ [Answer: 4]
(m) $\mathcal{P}(\{1\}) \times \mathcal{P}(\{1,2\})$ [Answer: 8]
(n) $\mathcal{P}(\{1\}) \times \mathcal{P}(\{1,2,3\})$ [Answer: 16]
(o) $\mathcal{P}(\{1,2,3\}) \times \mathcal{P}(\{1,2,3\})$ [Answer: 64]
(p) $\mathcal{P}(\{1,2,3\}) \times \mathcal{P}(\{\{1,2,3\}\})$ [Answer: 16]
(q) $\mathcal{P}(\emptyset) \times \mathcal{P}(\emptyset)$ [Answer: 1]

(r) $\mathcal{P}(\emptyset) \times \mathcal{P}(\{\{\emptyset\}\})$ [Answer: 2]

(s) $\mathcal{P}(\{\emptyset\}) \times \mathcal{P}(\{\emptyset\})$ [Answer: 4]

(t) $\mathcal{P}(\{\{\emptyset\}\}) \times \mathcal{P}(\{\{\emptyset\}\})$ [Answer: 4]

(u) $\mathcal{P}(\{\emptyset\}) \times \mathcal{P}(\{\emptyset, \{\emptyset\}\})$ [Answer: 8]

(v) $\mathcal{P}(\{\emptyset, \{\emptyset\}\}) \times \mathcal{P}(\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\})$ [Answer: 32]

1.7 Operations on Sets

1. Let $A = \{1, 2\}, B = \{2, 3\}, C = \{3, 4\}$ be sets in the universe $U = \{1, 2, 3, 4, 5\}$. Compute the following sets.

   (a) $A \cup C$ [Answer: $\{1, 2, 3, 4\}$]
   (b) $A \cap C$ [Answer: $\emptyset$]
   (c) $(A \cup C)^C$ [Answer: $\{5\}$]
   (d) $(A^C \cup C^C)$ [Answer: U]
   (e) $(A \cup B) \cap C$ [Answer: $\{3\}$]
   (f) $(A \cap B) \cup C$ [Answer: $\{2, 3, 4\}$]
   (g) $A \cup \emptyset$ [Answer: A]
   (h) $A \cap \emptyset$ [Answer: $\emptyset$]
   (i) $A \cap A$ [Answer: A]
   (j) $A \cup A$ [Answer: A]
   (k) $A - A$ [Answer: $\emptyset$]
   (l) $A - \emptyset$ [Answer: A]
   (m) $\emptyset - A$ [Answer: $\emptyset$]
   (n) $\emptyset - \emptyset$ [Answer: $\emptyset$]
   (o) $\emptyset \cap \emptyset$ [Answer: $\emptyset$]
   (p) $\emptyset \cup \emptyset$ [Answer: $\emptyset$]
   (q) A^C [Answer: $\{3, 4, 5\}$]
   (r) $(A^C)^C$ [Answer: A]
   (s) $(A - B) \cup (B - A)$ [Answer: $\{1, 3\}$]
   (t) $(A - B) \cap (B - A)$ [Answer: $\emptyset$]
   (u) U^C [Answer: $\emptyset$]
   (v) $A - U$ [Answer: $\emptyset$]
   (w) $U - A$ [Answer: $\{3, 4, 5\}$]

2. Let $A = \{2, 4, 6, 8, 10, 12, 14\}, B = \{3, 6, 9, 12, 15\}, C = \{5, 10, 15\}$ be sets in the universe $U = \{n \in \mathbb{N} : n \leq 15\}$. Compute the following sets.

   (a) $A \cup B$ [Answer: $\{2, 3, 4, 6, 9, 8, 10, 12, 14, 15\}$]
   (b) $A \cup C$ [Answer: $\{2, 4, 5, 6, 8, 10, 12, 14, 15\}$]
   (c) $B \cup C$ [Answer: $\{3, 5, 6, 9, 10, 12, 15\}$]
   (d) $A \cap B$ [Answer: $\{6, 12\}$]
   (e) $A \cap C$ [Answer: $\{10\}$]
   (f) $B \cap C$ [Answer: $\{15\}$]

(g) $A \cap B \cap C$ [Answer: $\emptyset$]

(h) $A \cap (B \cup C)$ [Answer: $\{6, 10, 12\}$]

(i) $(A \cap B) \cup C$ [Answer: $\{5, 6, 10, 12, 15\}$]

(j) $A \cup (B \cap C)$ [Answer: $\{2, 4, 6, 8, 10, 12, 14, 15\}$]

(k) $(A \cup B) \cap C$ [Answer: $\{10, 15\}$]

(l) $A - B$ [Answer: $\{2, 4, 8, 10, 14\}$]

(m) $B - A$ [Answer: $\{3, 9, 15\}$]

(n) $A - (B \cap C)$ [Answer: $A - \{15\} = A$]

(o) $(A - B) \cap C$ [Answer: $\{2, 4, 8, 10, 14\} \cap C = \{10\}$]

(p) $B - (A \cap C)$ [Answer: $B - \{10\} = B$]

(q) $(B - A) \cap C$ [Answer: $\{3, 9, 15\} \cap C = \{15\}$]

(r) $A - (B - C)$ [Answer: $A - \{3, 6, 9, 12\} = \{2, 4, 8, 10, 14\}$]

(s) $(A - B) - C$ [Answer: $\{2, 4, 8, 10, 14\} - C = \{2, 4, 8, 14\}$]

(t) A^C [Answer: $\{1, 3, 5, 7, 9, 11, 13, 15\}$]

(u) $A \cap A^C$ [Answer: $\emptyset$]

(v) $A \cup A^C$ [Answer: U]

(w) $A \cap B^C$ [Answer: $\{2, 4, 8, 10, 14\}$]

(x) $A \cup B^C$ [Answer: $\{U - \{3, 9, 15\}\}$]

(y) $A^C - C^C$ [Answer: $\{5, 15\}$]

(z) $(A - C)^C$ [Answer: $\{2, 4, 6, 8, 12, 14\}^C = \{1, 3, 5, 7, 9, 10, 11, 13, 15\}$]

3. Let $A_k = \{nk : n \in \mathbb{N}\}$. Compute the following sets. [Note: For example $A_1 = \mathbb{N}$, $A_2 = \{2, 4, 6, 8, \cdots\}$, $A_3 = \{3, 6, 9, \cdots\}$, $A_4 = \{4, 8, 12, \cdots\}$ and so on.]

(a) $A_1 \cap A_2$ [Answer: $\{2, 4, 6, \cdots\} = A_2$]

(b) $A_1 \cup A_3$ [Answer: $\{1, 2, 3, \cdots\} = A_1$]

(c) $A_2 \cap A_3$ [Answer: $\{6, 12, 18, \cdots\} = A_6$]

(d) $A_4 \cap A_2$ [Answer: $\{4, 8, 12, \cdots\} = A_4$]

(e) $A_3 \cup A_5$ [Answer: $\{3, 5, 6, 9, 10, 12, 15, 18, 20, 21, 24, 25, \cdots\}$]

(f) $A_1 \cup (A_2 \cup A_3)$ [Answer: $A_1 \cup \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, \cdots\} = A_1$]

(g) $(A_1 \cup A_2) \cup A_3$ [Answer: $A_1 \cup A_3 = A_1$]

(h) $A_1 \cap (A_2 \cap A_3)$ [Answer: $A_1 \cap A_6 = A_6$]

(i) $(A_1 \cap A_2) \cap A_3$ [Answer: $A_2 \cap A_3 = A_6$]

(j) $A_1 \cup (A_2 \cap A_3)$ [Answer: $A_1 \cup A_6 = A_1$]

(k) $(A_1 \cup A_2) \cap A_3$ [Answer: $A_1 \cap A_3 = A_3$]

(l) $A_{15} \cup (A_3 \cap A_5)$ [Answer: $A_{15} \cup A_{15} = \{15, 30, 45, \cdots\} = A_{15}$]

(m) $(A_{15} \cup A_3) \cap A_5$ [Answer: $A_3 \cap A_5 = \{15, 30, 45, \cdots\} = A_{15}$]
(n) $(A_2 \cup A_3) \cap A_4$ [Answer: $\{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, \cdots\} \cap A_4 = \{4, 8, 12, \cdots\} = A_4$]
(o) $A_2 \cup (A_3 \cap A_4)$ [Answer: $A_2 \cup A_{12} = \{2, 4, 6, \cdots\} = A_2$]
(p) $(A_4 \cap A_3) - A_6)$ [Answer: $A_{12} - A_6 = \emptyset$]
(q) $A_6 - (A_4 \cap A_3)$ [Answer: $A_6 - A_{12} = \{6, 18, 30, 42, \cdots\} = \{12n - 6 : n \in \mathbb{N}\}$]
(r) $(A_1 - A_2) \cap A_3$ [Answer: $\{1, 3, 5, 7, \cdots\} \cap A_3 = \{3, 9, 15, 21, \cdots\} = \{6n - 3 : n \in \mathbb{N}\}$]
(s) $A_1 - (A_2 \cap A_3)$ [Answer: $A_1 - A_6 = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, \cdots\} = \mathbb{Z} - A_6$]
(t) $(A_1 - A_2) - A_3$ [Answer: $\{1, 3, 5, 7, \cdots\} - A_3 = \{1, 5, 7, 11, 13, 17, 19, \cdots\}$]
(u) $A_1 - (A_2 - A_3)$ [Answer: $A_1 - A_6 = A_1 - \{2, 4, 8, 10, 14, 16, \cdots\} = \{1, 3, 5, 6, 7, 9, 11, 13, 14, 15, \cdots\}$]

4. Let $A_z = \{x \in \mathbb{R} : |x| \le z\}$, $B_z = \{x \in \mathbb{R} : |x| \ge z\}$ and compute the following sets. [Note: For example $A_2 = [-2, 2]$, $A_3 = [-3, 3], B_0 = \mathbb{R}, B_1 = (-\infty, -1] \cup [1, \infty), B_4 = (-\infty, -4] \cup [4, \infty)$ and so on.]

(a) $A_1 \cap A_2$ [Answer: A_1]
(b) $A_1 \cup A_2$ [Answer: A_2]
(c) $B_1 \cap B_2$ [Answer: B_2]
(d) $B_1 \cup B_2$ [Answer: B_1]
(e) $A_1 \cap B_1$ [Answer: $\{-1, 1\}$]
(f) $B_1 \cap \mathbb{N}$ [Answer: $\mathbb{N}$]
(g) $B_{3/2} \cap \mathbb{N}$ [Answer: $\mathbb{N} - \{1\}$]
(h) $A_1 \cap \mathbb{N}$ [Answer: $\{1\}$]
(i) $A_{3/2} \cap \mathbb{N}$ [Answer: $\{1\}$]
(j) $A_1 \cap B_2$ [Answer: $\emptyset$]
(k) $A_2 \cap B_1$ [Answer: $[-2, 1] \cup [1, 2]$]
(l) $B_1 - B_2$ [Answer: $(-2, 1] \cup [1, 2)$]
(m) $B_2 - B_1$ [Answer: $\emptyset$]
(n) $A_1 - A_2$ [Answer: $\emptyset$]
(o) $A_2 - A_1$ [Answer: $[-2, 1) \cup (1, 2]$]
(p) $A_1 - B_1$ [Answer: $(-1, 1)$]
(q) $B_1 - A_1$ [Answer: $(-\infty, -1) \cup (1, \infty)$]

1.8 Set Builder Notation

Note that there are many ways to express things in English sentences and also in set builder notation, so each of these has many different answers. Only a few possibilities are included.

1. Translate the following sets described in set builder notation into English sentences. As there are many ways to say the same thing in English, there will be many answers to each of these questions. Here we list only one.

    (a) $\{2n : n \in \mathbb{Z}\}$ [Answer: The even numbers.]

    (b) $\{2n+2 : n \in \mathbb{Z}\}$ [Answer: The even numbers.]

    (c) $\{n \in \mathbb{Z} : \frac{n}{2} \in \mathbb{Z}\}$ [Answer: The even numbers.]

    (d) $\{n \in \mathbb{Z} : (\frac{n}{4} \in \mathbb{Z}) \vee (\frac{n+2}{4} \in \mathbb{Z})\}$ [Answer: The even numbers.]

    (e) $\{n : \frac{n^2}{4} \in \mathbb{Z}\}$ [Answer: The even numbers.]

    (f) $\{2n : n \in \mathbb{N}\}$ [Answer: The positive even numbers.]

    (g) $\{2n-1 : n \in \mathbb{N}\}$ [Answer: The positive odd numbers.]

    (h) $\{2n+3 : n \in \mathbb{Z}\}$ [Answer: The odd numbers.]

    (i) $\{2n-5 : n \in \mathbb{Z}\}$ [Answer: The odd numbers.]

    (j) $\{n^2 : n \in \mathbb{Z}\}$ [Answer: The set of perfect squares.]

    (k) $\{(2n)^2 : n \in \mathbb{N}\}$ [Answer: The set of even perfect squares.]

    (l) $\{(2n-1)^2 : n \in \mathbb{N}\}$ [Answer: The set of odd perfect squares.]

    (m) $\{(2n+1)^2 : n \in \mathbb{N}\}$ [Answer: The set of odd perfect squares.]

    (n) $\{2^n : n \in \mathbb{N}\}$ [Answer: The set of powers of two that are bigger than one.]

    (o) $\{n \in \mathbb{Z} : |n| \neq n\}$ [Answer: The negative integers.]

    (p) $\{n \in \mathbb{Z} : \sqrt{n^2} \neq n\}$ [Answer: The negative integers.]

    (q) $\{n \in \mathbb{Z} : \sqrt{n^2} = n\}$ [Answer: The set of non-negative integers.]

    (r) $\{n \in \mathbb{Z} : \sqrt[3]{n^3} = n\}$ [Answer: The integers.]

    (s) $\{n \in \mathbb{Z} : |n| \geq n\}$ [Answer: The integers.]

    (t) $\{n \in \mathbb{Z} : |n| > n\}$ [Answer: The negative integers.]

    (u) $\{n+3 : n \in \mathbb{Z}\}$ [Answer: The integers.]

    (v) $\{2n : (\exists m \in \mathbb{Z})\ n = 3m\}$ [The multiples of six.]

    (w) $\{3n : (\exists m \in \mathbb{Z})\ n = 2m\}$ [The multiples of six.]

    (x) $\{n \in \mathbb{Z} : (\frac{n}{2} \in \mathbb{Z}) \wedge (\frac{n}{3} \in \mathbb{Z})\}$ [The multiples of six.]

    (y) $\{6n - 216 : n \in \mathbb{Z}\}$ [The multiples of six.]

    (z) $\{216 - 6n : n \in \mathbb{Z}\}$ [The multiples of six.]

2. Translate the following sets described in set builder notation into listed form.

(a) $\{2n+1 : n \in \mathbb{N}\}$ [Answer: $\{3, 5, 7, 9, \cdots\}$.]
(b) $\{n-1 : n \in \mathbb{N}\}$ [Answer: $\{0, 1, 2, 3, 4, 5, \cdots\}$.]
(c) $\{(-1)^n : n \in \mathbb{N}\}$] [Answer: $\{-1, 1\}$.]
(d) $\{(-1)^{n+1} : n \in \mathbb{N}\}$] [Answer: $\{-1, 1\}$.]
(e) $\{(-1)^n : n \in \mathbb{Z}\}$] [Answer: $\{-1, 1\}$.]
(f) $\{\cos(\pi n) : n \in \mathbb{N}\}$] [Answer: $\{-1, 1\}$.]
(g) $\{n \in \mathbb{Z} : n^2 = 1\}$] [Answer: $\{-1, 1\}$.]
(h) $\{n \in \mathbb{Q} : n^2 = 1\}$] [Answer: $\{-1, 1\}$.]
(i) $\{n \in \mathbb{R} : n^2 = 1\}$] [Answer: $\{-1, 1\}$.]
(j) $\{n \in \mathbb{R} : \frac{n^3-n}{n} = 0\}$] [Answer: $\{-1, 1\}$.]
(k) $\{3(n-1) : n \in \mathbb{N}\}$ [Answer: $\{0, 3, 6, 9, 12, 15, \cdots\}$.]
(l) $\{3n-2 : n \in \mathbb{N}\}$ [Answer: $\{1, 4, 7, 10, 13, 16, \cdots\}$.]
(m) $\{3n+1 : n \in \mathbb{N}\}$ [Answer: $\{4, 7, 10, 13, 16, \cdots\}$.]
(n) $\{3n-2 : n \in \mathbb{Z}\}$ [Answer: $\{\cdots, -8, -5, -2, 1, 4, 7, 10, 13, 16, \cdots\}$.]
(o) $\{3n : n \in \mathbb{Z}\}$ [Answer: $\{\cdots, -9, -6, -3, 0, 3, 6, 9, \cdots\}$.]
(p) $\{4n+1 : n \in \mathbb{N}\}$ [Answer: $\{5, 9, 13, 17, 21, \cdots\}$.]
(q) $\{4n+1 : n \in \mathbb{Z}\}$ [Answer: $\{\cdots, -7, -3, 1, 5, 9, 13, 17, 21, \cdots\}$.]
(r) $\{4n+3 : n \in \mathbb{Z}\}$ [Answer: $\{\cdots, -9, -5, -1, 3, 7, 11, 15, \cdots\}$.]
(s) $\{2n-5 : n \in \mathbb{Z}\}$ [Answer: $\{\cdots, -5, -3, -1, 1, 3, 5, \cdots\}$.]
(t) $\{n^2(-1)^{n+1} : n \in \mathbb{N}\}$] [Answer: $\{1, -4, 9, -16, 25, -36, 49, \cdots\}$.]
(u) $\{\frac{1}{2n} : n \in \mathbb{N}\}$] [Answer: $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10} \cdots\}$.]
(v) $\{(\frac{1}{2})^n : n \in \mathbb{N}\}$] [Answer: $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32} \cdots\}$.]
(w) $\{\frac{1}{2^n} : n \in \mathbb{N}\}$] [Answer: $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32} \cdots\}$.]
(x) $\{2^{-n} : n \in \mathbb{N}\}$] [Answer: $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32} \cdots\}$.]
(y) $\{(-2)^{-n} : n \in \mathbb{N}\}$] [Answer: $\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32} \cdots\}$.]
(z) $\{\frac{1}{n^2} : n \in \mathbb{N}\}$] [Answer: $\{\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25} \cdots\}$.]

3. Translate the following sets described in English sentences into ones using only set builder notation and mathematical symbols. There are many correct ways to answer each of these questions.

   (a) The set of multiples of three. [Answer: $\{3n : n \in \mathbb{Z}\}$ or $\{n \in \mathbb{Z} : \frac{n}{3} \in \mathbb{Z}\}$ or $\{3n+3 : n \in \mathbb{Z}\}$.]
   (b) The set of even numbers.
   (c) The set of multiples of six. [Answer: $\{6n : n \in \mathbb{Z}\}$ or $\{n \in \mathbb{Z} : 6 \mid n\}$ or $\{n \in \mathbb{Z} : (2 \mid n) \wedge (3 \mid n)\}$.]
   (d) The set of odd numbers. [Answer: $\{2n+1 : n \in \mathbb{Z}\}$ or $\{2n-1 : n \in \mathbb{Z}\}$ or $\{2n+3 : n \in \mathbb{Z}\}$ or $\{2n-3 : n \in \mathbb{Z}\}$ or $\{2n+101 : n \in \mathbb{Z}\}$ or $\{n \in \mathbb{Z} : \frac{2n+1}{2} \in \mathbb{Z}\}$ or $\{n \in \mathbb{Z} : \frac{2n-1}{2} \in \mathbb{Z}\}$ $\{n \in \mathbb{Z} : \frac{n}{2} \notin \mathbb{Z}\}$.]
   (e) The set of numbers with a remainder of one when divided by three.

(f) The set of numbers with a remainder of two when divided by four. [Answer: $\{n \in \mathbb{Z} : \frac{n+2}{4} \in \mathbb{Z}\}$ or $\{n \in \mathbb{Z} : \frac{n-2}{4} \in \mathbb{Z}\}$ or $\{4n+2 : n \in \mathbb{Z}\}$ or $\{4n-2 : n \in \mathbb{Z}\}$.]

(g) The set of perfect squares.

(h) The set of positive perfect squares. [Answer: $\{n^2 : n \in \mathbb{N}\}$ or $\{n : (\exists m \in \mathbb{N})\ n = m^2\}$]

(i) The set of perfect cubes. [Answer: $\{n^3 : n \in \mathbb{Z}\}$ or $\{n : (\exists m \in \mathbb{Z})\ n = m^2\}$]

(j) The set of positive perfect cubes.

(k) The set of negative perfect cubes. [Answer: $\{n^3 : (n \in \mathbb{Z}) \wedge (n < 0)\}$ or$\{-(n^3) : n \in \mathbb{N}\}$ or $\{(-n)^3 : n \in \mathbb{N}\}$]

(l) The set of whole numbers bigger than two.

(m) The set of whole numbers bigger than three. [Answer: $\{n \in \mathbb{Z} : n > 3\}$ or $\{n \in \mathbb{Z} : n \geq 4\}$ or $\{n \in \mathbb{N} : n^2 > 9\}$ or $\{n \in \mathbb{N} : n^2 > 10\}$ or $\{n \in \mathbb{N} : n^2 \geq 16\}$ or $\{n \in \mathbb{N} : 2n \geq 7\}$ or $\{n \in \mathbb{Z} : 2n \geq 7\}$.]

(n) The set of numbers strictly between -2 and 2. [Answer: $\{n \in \mathbb{Z} : -2 < n < 2\}$ or $\{n \in \mathbb{Z} : -1 \leq n \leq 1\}$ or$\{x \in \mathbb{R} : x^3 = x\}$.]

(o) The set of positive integers with decimal expansions ending in one. [Answer: $\{10n+1 : n \in \mathbb{Z} \wedge n \geq 0\}$ or $\{10n-9 : n \in \mathbb{N}\}$.]

(p) The set of positive integers with decimal expansions ending in zero. [Answer: $\{10n : n \in \mathbb{N}\}$.]

(q) The set of positive integers with decimal expansions ending in five. [Answer: $\{10n-5 : n \in \mathbb{N}\}$ or $\{5n : (n \in \mathbb{N}) \wedge (\frac{n}{2} \notin \mathbb{Z})\}$ or $\{n : (\frac{n}{5} \in \mathbb{Z}) \wedge (\frac{n}{2} \notin \mathbb{Z})\}$.]

(r) The set of positive rational numbers expressible with a one in the numerator. [Answer: $\{\frac{1}{n} : n \in \mathbb{N}\}$.]

(s) The set of rational numbers expressible as a fraction with a one in the numerator. [Answer: $\{\frac{1}{n} : n \in \mathbb{Z} - \{0\}\}$.]

4. Translate the following sets in listed form into ones using only set builder notation and mathematical symbols. There are many correct ways to answer each of these questions.

(a) $\{7, 14, 21, 28, 35, 42 \cdots\}$

(b) $\{\cdots, -35, -28, -21, -14, -7, 0, 7, 14, 21, 28, 35 \cdots\}$

(c) $\{1, 4, 7, 10, 13, 16, 19 \cdots\}$ [Answer: $\{3n-2 : n \in \mathbb{N}\}$.]

(d) $\{7, 10, 13, 16, 19, 22, 25 \cdots\}$ [Answer: $\{3n+4 : n \in \mathbb{N}\}$.]

(e) $\{2, 5, 8, 11, 14, 17 \cdots\}$

(f) $\{-1, 2, 5, 8, 11, 14, \cdots\}$

(g) $\{-3, -2, -1, 0, 1, 2, 3, \cdots\}$ [Answer: $\{n-4 : n \in \mathbb{N}\}$ or $\{n \in \mathbb{Z} : n \geq -3\}$ or $\{n \in \mathbb{Z} : n > -4\}$.]

(h) $\{1, -2, 3, -4, 5, -6, \cdots\}$ [Answer: $\{n(-1)^{n+1} : n \in \mathbb{N}\}$ or $\{-n(-1)^n : n \in \mathbb{N}\}$.]

(i) $\{-2, 4, -6, 8, -10, 12, \cdots\}$

(j) $\{-1, -4, -9, -16, -25, -36, \cdots\}$ [Answer: $\{-n^2 : n \in \mathbb{N}\}$.]

(k) $\{0, 1, 4, 9, 16, \cdots\}$

(l) $\{9, 16, 25, 36, 49, \cdots\}$

(m) $\{5, 9, 13, 17, 21, \cdots\}$

(n) $\{\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{12}, \frac{1}{15}, \cdots\}$

(o) $\{\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \cdots\}$

(p) $\{-\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, -\frac{16}{81}, \cdots\}$ [Answer: $\{(-\frac{2}{3})^n : n \in \mathbb{N}\}$.]

(q) $\{2, 1, \frac{2}{3}, \frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \frac{2}{7}, \frac{1}{8}, \cdots\}$ [Answer: $\{\frac{2}{n} : n \in \mathbb{N}$.]

(r) $\{9, 99, 999, 9999, 99999, \cdots\}$ [Answer: $\{10^n - 1 : n \in \mathbb{N}\}$.]

(s) $\{11, 101, 1001, 10001, \cdots\}$ [Answer: $\{10^n + 1 : n \in \mathbb{N}\}$.]

(t) $\{101, 10001, 100001, 1000001, \cdots\}$ [Answer: $\{100^n + 1 : n \in \mathbb{N}\}$.]

(u) $\{101, 1001, 10001, 100001, \cdots\}$ [Answer: $\{10^n + 1 : (n \in \mathbb{N}) \wedge (n \geq 2)\}$ or $\{10^{n+1} + 1 : n \in \mathbb{N}\}$ or $\{10 \cdot 10^n + 1 : n \in \mathbb{N}\}$.]

Chapter 2

Basic Proof Techniques

2.1 Parity Proofs

Prove each of the following statements for the integers a, b and c. You may assume that the integers are closed under addition and multiplication, that any integer is even or odd, and that one is not an even integer.

1. If a is even then $3a$ is even.

   Proof: Suppose that a is an even number. We know then that $a = 2s$ for some s in $\mathbb{Z}$. We can see that $3a = 3 \cdot 2s = 2 \cdot 3s$. Set t to be the integer $3s$. Then $3a = 2t$ for $t \in \mathbb{Z}$ and by definition we get that $3a$ is even. $\square$

2. If a is odd then $5a$ is odd.

   Proof: Let a be an odd number. By definition, $a = 2s + 1$ for some s in the integers. Then $5a = 5(2s + 1) = 10s + 5 = 10s + 4 + 1 = 2(5s + 2) + 1$. Let t be the integer $5s + 2$. Now $5a = 2t + 1$ and by definition $5a$ is odd. $\square$

3. If a is even then $-a$ is even.

4. If a is even then $3a$ is even.

5. If a is odd then $-a$ is odd.

   Proof: Suppose a is an odd number. Thus a must equal $2k+1$ for some integer k. then $-a = -(2k+1) = -2k - 1$. We can "add zero" to get $-a = -2k - 1 + (-1 + 1) = (-2k - 2) + 1 = 2(-k - 1) + 1$. Setting $m = -k - 1$ which is an integer, gives $-a = 2m + 1$ for $m \in \mathbb{Z}$, which proves that $-a$ is odd. $\square$

6. If a is even then $a - 1$ is odd.

   Proof: Assume a is even so $a = 2r$ for $r \in \mathbb{Z}$. Then $a - 1 = 2r - 1 = 2r - 2 + 1 = 2(r - 1) + 1$. Set $s = r - 1$ which is an integer to see that $a - 1 = 2s + 1$ and therefore is odd.

7. If a is odd then $a - 4$ is odd.

8. If $a+5$ is even then a is odd.

   Proof: Assume that $a+5$ is even, thus we know $a+5=2s$ for $s \in \mathbb{Z}$. Then $a=2s-5=2s-6+1=2(s-3)+1$. Set $t=s-3$ which is in $\mathbb{Z}$. We then get $a=2t+1$ which shows that a is odd. $\square$

   Alternate Proof: We can take the contrapositive and instead show that if a is even then $a+5$ is odd. Assume $a=2s$ for $s \in \mathbb{Z}$. Then $a+5=2s+5=2s+4+1=2(s+2)+1$. Set $t=s+2$. Then $a+5=2t+1$, and as t is an integer, this shows $a+5$ is odd. $\square$

9. If a is odd then $a+5$ is even.

10. If a is even then a^2 is even.

11. If a is odd then a^3 is odd.

    Proof: Suppose a is odd so $a=2k+1$ for $k \in \mathbb{Z}$. Thus $a^3=(2k+1)^3=8k^3+12k^2+6k+1=2(4k^3+6k^2+3k)+1$. Set $m=4k^3+6k^2+3k \in \mathbb{Z}$. Now $a^3=2m+1$ and therefore is odd. $\square$

12. If a^2 is odd then a is odd.

13. If a^2 is even then a is even.

14. If a^3 is odd then a is odd.

    Proof: We take the contrapositive and instead show that if a is even then so is a^3. Assume $a=2k$. Then $a^3=8k^3=2(4k^3)$. Set $l=4k^3 \in \mathbb{Z}$ to get $a^3=2l$ and show it is even. $\square$

15. If a^5+8 is odd then a is odd.

16. If a^3-3a^2+a is even then a is even.

    Proof: We instead show the contrapositive of the original statement. Assume that a is odd and thus $a=2r+1$ for some $r \in \mathbb{Z}$. Then $a^3-3a^2+a=(2r+1)^3-3(2r+1)^2+2r+1=(2r+1)(4r^2+4r+1)-3(4r^2+4r+1)+2r+1=(8r^3+12r^2+3r+1)-12r^2-12r-3+2r+1=8r^3-4r-1=8r^3-4r-2+1=2(4r^3-2r-1)+1$. Set $s=4r^3-2r-1$ to see that $a^3-3a^2+a=2s+1$. Because s is an integer, this shows a^3-3a^2+a is odd. $\square$

17. If a and b are even then their sum is even.

18. If a and b are even then the product ab is even.

    Proof: Assume that a and b are both even. Thus $a=2r$ and $b=2s$ for some r and s in the integers. Then $ab=2r \cdot 2s=2 \cdot (2rs)$. Set t to be the integer $2rs$. Then $ab=2t$ for $t \in \mathbb{Z}$ thus showing ab is even. $\square$

19. If a and b are odd then their sum is even.

20. If a and b are odd then their difference is even.

21. If a and b are odd then their product is odd.

22. If a is even and b is odd then the product ab is even.

    Proof: Assume that $a=2k$ and $b=2l+1$ for $k,l \in \mathbb{Z}$. then $ab=(2k)(2l+1)=2(2lk+k)$. Let $m=2lk+k$. Then $ab=2m$ and as $m \in \mathbb{Z}$ this shows ab is even. $\square$

23. If a is even and b is odd then the sum $a+b$ is odd.

24. If $b-a$ is odd then $b+a$ is odd.

    Proof: Suppose $b-a$ is odd, then $b-a=2r+1$ for some r. Then as $b=2r+a+1$ we know that $b+a=(2r+a+1)+a=2r+2a+1=2(r+a)+1$. Set s to be the integer $r+a$. We now have $b+a=2s+1$ which shows it to be odd. □

25. If $b-a$ is even then $b+a$ is even.

26. If $b+2a$ is even then ba is even.

27. If $b+2a$ is even then b^3a is even.

    Proof: Suppose $b+2a$ is even. Then $b+2a=2r$ for some integer r. This means $b=2a-2r=2(a-r)$ and therefore $b^3a=(2(a-r))^3a=8(a-r)^3a=2(4(a-r)^3a)$. We set s to be the integer $(4(a-r)^3a)$ to get $b^3a=2s$, thus showing it is even. □

28. If $4b-a$ is odd then a^2-16b^2 is odd.

    Proof: Suppose $4b-a$ is odd. Then $4b-a=2k+1$ for some integer k. Thus $a=4b-2k-1$ which implies $a^2-16b^2=(4b-2k-1)(4b-2k-1)-4b=16b^2-16bk-8b+4k^2+4k+1-16b^2=-16bk-8b+4k^2+4k+1=2(-8bk-4b-2k^2-2k)+1$. If we set m to be the integer $-8bk-4b-2k^2-2k$ we see that $a^2-16b^2=2m+1$ and therefore is odd. □

29. If a is even or b is even then their product is even.

    Proof: We have two different cases, but since ab is equal ba we can, without loss of generality, make the assumption that a is even. Assume $a=2k$ for $k\in\mathbb{Z}$. Then $ab=2kb$. Letting $l=kb\in\mathbb{Z}$ we get that the product is $2l$, which shows it is even. □

30. If $a+1$ is odd or $b-1$ is odd then their product is even.

    Proof: We have two different cases, and here we can't just switch which is which, since we know different facts about a and b. Therefore we need to split this into separate cases.

    Case 1: Assume $a+1$ is odd. Here $a+1=2r+1$ for $r\in\mathbb{Z}$ and thus $a=2r$. Then $ab=2(rb)$. Set s to be the integer rb to get that $ab=2s$ and see that this product is odd.

    Case 2: Assume that $b-1$ is odd. Thus $b-1=2t+1$[1] for $t\in\mathbb{Z}$. Then $b=2t+2=2(t+1)$ and thus $ab=2a(t+1)$. Setting $u=a(t+1)$ we get that $ab=2u$. Since u is an integer, this shows that ab is even. □

31. If ab is odd then a and b are odd.

32. For any a, $4a$ is even.

    Proof: Whatever a is we can set $k=2a\in\mathbb{Z}$ to write $4a=2k\in\mathbb{Z}$ which shows it is even.[2] □

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[1] Since each case is a separate proof, we don't really need to use new letters here. We could have reused r and s at this point if we really wanted to.

[2] We could have also broken this down into two cases for a odd, and a even.

33. For any integer a, $a^2 + a$ is always even.

    Proof: We can't pull out a two this time so we proceed by cases with the two possibilities that a is either even or odd.

    Case 1: a is even. Assume $a = 2k$ for some integer k. Then $a^2 + a = (2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k)$. Setting $l = 2k^2 + k \in \mathbb{Z}$ gives us that $a^2 + a = 2l$ and thus is even.

    Case 2: a is odd. Assume $a = 2k + 1$ for an integer value of k. Here $a^2 + a = (2k + 1)^2 + 2k + 1 = 4k^2 + 4k + 1 + 2k + 1 = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$. By letting l be the integer $2k^2 + 3k + 1$ we get that $a^2 + a = 2l$ showing it to be even. $\square$

34. For any integer a, $a^2 - a - 1$ is always odd.

35. For any integer a, $a^3 + a^2$ is always even.

36. For any integer a, $a(a + 1)$ is even.

37. If a is odd then $\frac{a^2-1}{4}$ is an even integer. [Hint: Use the last result.]

    Proof: Assume a is odd. Then $a = 2k + 1$ and $\frac{a^2-1}{4} = \frac{(2k+1)^2-1}{4} = \frac{4k^2+4k+1-1}{4} = \frac{4k^2+4k}{4} = k^2 + k = k(k + 1)$. By our last result this is even, completing our proof. $\square$

38. For any integer a, $a^2 - a - 6$ is even.

    Proof: We split this into cases.

    Case 1: a is odd. Here $n = 2k + 1$ for $k \in \mathbb{Z}$. Thus $a^2 - a - 6 = (2k + 1)^2 - (2k + 1) - 6 = 4k^2 + 4k + 1 - 2k - 1 - 6 = 4k^2 + 2k - 6 = 2(2k^2 + k - 3)$. Setting $m = 2k^2 + k - 3 \in \mathbb{Z}$ shows that $a^2 - a - 6 = 2m$ and thus is even.

    Case 2: a is even. Here $a = 2k$ for $k \in \mathbb{Z}$. Thus $a^2 - a - 6 = (2k)^2 - (2k) - 6 = 4k^2 - 2k - 6 = 2(2k^2 - k - 3)$. Setting $m = 2k^2 - k - 3 \in \mathbb{Z}$ shows that $a^2 - a - 6 = 2m$ and thus is even.

39. If a or b is even then $ab - 3$ is odd.

40. If a or b is odd then $ab + b + a$ is odd.

    Proof: Without loss of generality, assume a is odd. Then $a = 2k + 1$ for some $k \in \mathbb{Z}$. Thus $ab + b + a = (2k + 1)b + b + (2k + 1) = 2kb + 2b + 2k + 1 = 2(kb + b + k) + 1$. Set $l = kb + b + k$ which is in $\mathbb{Z}$. Then $ab + b + a = 2l + 1$, completing our proof. $\square$

41. If a or b is even then $ab + 2b + 1$ is odd.

    Note: Here a and b do not play the same role in the expression we are looking at, so we can't do both cases at once.

    Proof:

    Case 1: a is even. Here $a = 2r$ for some $r \in \mathbb{Z}$. Thus $ab + 2b + 1 = 2rb + 2b + 1 = 2(rb + b) + 1$. By setting $s = rb + b \in \mathbb{Z}$ we have $ab + 2b + 1 = 2s + 1$ and thus see it is odd.

    Case 2: b is even. Here $b = 2r$ for some $r \in \mathbb{Z}$. Thus $ab + 2b + 1 = 2ar + 4r + 1 = 2(ar + 2r) + 1$. We can then set s equal to $ar + 2r$ which is an integer to write $ab + 2b + 1$ as $2s + 1$ showing it is odd. $\square$

42. If a or b is even then $(ab)^2$ is even.

43. If a is even then either $ab-b$ or $ab-b-a-1$ is odd.

    Proof: We suppose that a is even and $ab-b$ is not odd and then will show that $ab-b-a-1$ is odd. We know $a=2r$ and $ab-b=2s$ for $r,s\in\mathbb{Z}$. Then $ab-b-a-1=2s-2r-1=2s-2r-2+1=2(s-r-1)+1$. Setting $t=s-r-1\in\mathbb{Z}$ we see $ab-b-a-1$ is odd. □

44. If ab is even then a or b is even.

45. If $a+b$ is odd then a or b is odd.

46. If a is odd and b is any integer, then either ab or $ab+a$ is odd.

47. If a is even and b is any integer, then either $ab+b$ or $ab+a+b+1$ is odd.

    Proof: Assume that a is even and that $ab+b$ is not odd and we will show $ab+a+b+1$ is odd. We can assume that $a=2k$ and that $ab+b=2m$ for some integers k and m. Then $ab+a+b+1=(ab+b)+a+1=2m+2k+1=2(m+k)+1$. If we set n to be the integer $m+k$ we get $ab+a+b+1=2n+1$ which shows it is odd. □

48. If a is odd and b is any integer, then either ab or $ab+b$ is odd.

49. If $a+b$ is even then a and b have the same parity.

    Proof: We show the contrapositive statement: If a and b have different parity then $a+b$ is odd. We therefore know one of a and b is even and the other is odd. As a and b are interchangeable in the expression $a+b$ we can assume without loss of generality that a is even and b is odd. We know $a=2k$ for some integer k and $b=2m+1$ for some integer m. Then $a+b=2k+2m+1=2(k+m)+1$. Setting t to be the integer $k+m$ gives us $a+b=2t+1$ which shows it to be odd. □

50. If $b-a^2$ is even then a and b have the same parity.

    Proof: We prove the contrapositive statement: If a and b have different parity then $b-a^2$ is odd. As a and b are not interchangeable int he expression $b-a^2$, we must split this up into cases.

    Case 1: a is odd, b is even. Here $a=2r+1$ and $b=2s$ for some $r,s\in\mathbb{Z}$. Then $b-a^2=2s-(2r+1)^2=2s-(4r^2+4r+1)=2s-4r^2-4r-1=2s-4r^2-4r-1-1+1=2(s-2r^2-2r-1)+1$. Set $t=s-2r^2-2r-1\in\mathbb{Z}$ to get $b-2a^2=2t+1$ which shows that it is odd.

    Case 2: b is odd, a is even. Here $b=2r+1$ and $a=2s$ for some $r,s\in\mathbb{Z}$. Then $b-a^2=2r+1-(2s)^2=2r+1-4s^2=2(r-2s^2)+1$. We can set t to be the integer $r-2s^2$ to get $b-a^2=2t+1$, showing it to be odd. □

51. If $a+b$ is odd then a and b have different parity.

52. If $b-a^2$ is odd then a and b have different parity.

53. If a and b have the same parity then $a+b$ is even.

    Proof: If a and b have the same parity then they are either both odd or both even. We get the following cases.

    Case 1: a and b are odd. Here $a=2k+1$ and $b=2m+1$ for some integers k and m. Then $a+b=2k+1+2m+1=2k+2m+2=2(k+m+1)$. Set $n=k+m+1$ to get $a+b=2n$. Since n is an integer, this shows $a+b$ to be even.

    Case 2: a and b are even. Here $a=2k$ and $b=2m$ for some integers k and m. Then $a+b=2k+2m=2k+2m=2(k+m)$. Set $n=k+m$ to get $a+b=2n$. Since n is an integer, we see that $a+b$ is even. □

54. If a and b have the same parity then $b - a^2$ is even.

55. If a and b have different parity then $a + b$ is odd.

    Proof: Suppose a and b have different parity and we will show that $a+b$ is odd. We know one of a and b is even and the other is odd. As a and b are interchangeable in the expression $a + b$ we can assume a is even and b is odd without loss of generality. Thus $a = 2r$ and $b = 2s + 1$ for some integers r and s. Now $a + b = 2r + 2s + 1 = 2(r + s) + 1$. Setting t to be the integer $r + s$ we get that $a + b = 2t + 1$ which shows it to be an odd number. □

56. If a and b have different parity then $b - a^2$ is odd.

57. If a and b have different parity then ab is even.

58. If abc is even then a or b or c is even.

59. If abc is odd then a and b and c is odd.

    Proof: We prove the contrapositive statement: If a or b or c is even then abc is even. As a, b and c are interchangeable in the expression abc, we can assume that a is even without loss of generality. Then $a = 2r$ for some integer r which means $abc = 2rbc$. Setting s to be the integer rbc gives us $abc = 2s$ thus showing abc is even. □

60. If $a + b + c$ is odd then a or b or c is odd.

61. If $a + b + c$ is odd then $a + b$ or $b + c$ or $a + c$ is even.

    Proof: We prove the contrapositive statement: If $a + b$ and $b + c$ and $a + c$ are odd then $a + b + c$ is even. Assume $a + b$ and $b + c$ and $a + c$ are odd, thus we can write $a + b = 2r + 1$, $b + c = 2s + 1$ and $a + c = 2t + 1$ for some r, s and t in the integers. Then $a + b + c = 2r + 1 + c =$

62. Prove that if $a + b$ and $b + c$ are odd then $a + c$ is even.

    Proof: Assume that $a + b$ and $b + c$ are odd. Then $a + b = 2r + 1$ and $b + c = 2s + 1$ for some r and s in the integers. Then $a + c = 2r + 1 - b + 2s + 1 - b = 2r + 2 - 2s - 2b = 2(r + 1 - s - b)$. Set $t = r + 1 - s - b$, which is an integer, to get $a + c = 2t$. This shows $a + c$ is even. □

63. If $a + b + c$ is even then a or b or c is even.

    Proof: We prove the contrapositive statement: If a and b and c are odd then $a + b + c$ is odd. Assume that $a = 2r+1, b = 2s+1$ and $c = 2t+1$ for integers r, s and t. Then $a+b+c = 2r+1+2s+1+2t+1 = 2r+2s+2t+2+1 = 2(r+s+t+1)+1$. We can set u to be the integer $r+s+t+1$ to get $a+b+c = 2u+1$, which shows it to be odd. □

64. If $a + b + c$ is even then $a - b + c$ is even.

65. If $a + b + c$ is even then $a - b - c$ is even and $a + b - c$ is even.

    Proof: Suppose $a + b + c = 2k$. We have two statements we need to show. First $a = 2k - b - c$ so $a-b-c = 2k-b-c-b-c = 2k-2b-2c = 2(k-b-c)$. Setting $m = k-b-c \in \mathbb{Z}$ shows $a-b-c = 2m$ and thus is even. Next $a + b - c = 2k - b - c + b - c = 2k - 2c = 2(k - c)$. Setting $n = k - c$ shows $a + b - c = 2n$ which proves $a + b - c$ is even as n is an integer.[3] □

[3] In this argument, we could have instead used the fact that $a - b - c = a + b + c - 2b - 2c$ and $a + b - c = a + b + c - 2c$ which would have saved us a couple of steps.

66. No integer can be both odd and even.

Proof: We suppose such an integer does exist, and will reach a contradiction.[4] Suppose a is both even and odd. Then $a = 2k$ and $a = 2m+1$ for integers k and m. Then $2k = 2m+1$ so $1 = 2k-2m = 2(k-m)$. As $k - m$ is an integer, this shows that one is an even number, which is a contradiction. $\square$

67. For any integer a, a and $a + 1$ have different parity.

68. For any integer a, a and $-a$ have the same parity.

69. For any integer a, a and a^2 have the same parity.

70. For any integer a, a and $7a$ have the same parity.

Proof: Any integer a is either odd or even, thus we can break things down into two cases.

Case 1: a is even. Here $a = 2k$ for some integer k. Then $7a = 7(2k) = 2(7k)$. Setting $l = 7k \in \mathbb{Z}$ shows that $7k$ is even. We now know both a and $7a$ are even, thus they have the same parity.

Case 2: a is odd. Here $a = 2k + 1$ for $k \in \mathbb{Z}$. Then $7a = 7(2k + 1) = 14k + 7 = 14k + 7 - 1 + 1 = 14k + 6 + 1 = 2(7k + 3) + 1$. Set l to be the integer $7k + 3$ to show that $7k = 2l + 1$. This shows that $7a$ is odd and therefore both a and $7a$ have the same parity. $\square$

Alternate Proof: Suppose that a and $7a$ have different parity and we will reach a contradiction.

Case 1: a is even and $7a$ is odd. Here $a = 2r$ and $7a = 2s + 1$ for integers r and s. Then $2r = 2s + 1$ so $2(r - s) = 1$. As $r - s$ is an integer, this implies 1 is even, which is a contradiction.

Case 2: a is odd and $7a$ is even. Here $a = 2r + 1$ and $7a = 2s$ for integers r and s. Then $2r + 1 = 2s$ so $2(s - r) = 1$ which implies 1 is even because $s - r$ is an integer. This is a contradiction. $\square$

Disprove the following statements.

1. "If $6a$ is even then a is even."

Answer: The statement is equivalent to "$(\forall a)(6a \text{ even}) \Rightarrow (a \text{ even})$." We wish to show the negation is true. That statement is equivalent to "$\sim (\forall a)(6a \text{ even}) \Rightarrow (a \text{ even})$" or "$\sim (\forall a) \sim (6a \text{ even}) \vee (a \text{ even})$" or "$(\exists a) \sim\sim (6a \text{ even}) \wedge \sim (a \text{ even})$" or "$(\exists a)(6a \text{ even}) \wedge (a \text{ odd})$."

Thus, we only need to show one example where a is odd and $6a$ is even. We choose $a = 1$ (or any other odd) and we have accomplished that.

Though it's useful to use our rules for quantifiers to be sure we understand what is being stated, we can also try to figure things out without writing things in terms of quantifiers. If the original statement says that something is true for every a, then in order to show it is incorrect we need to find one a for which it isn't true.

2. "If a is odd then $a^2 + a$ is odd."

3. "If a is even then $\frac{a}{2}$ is even."

4. "If a is even then $\frac{a^2}{4}$ is even."

Answer: If $a = 2$ then then $\frac{a^2}{4}$ is one, which is not even. $\square$

[4] This is the only proof on this sheet that requires a proof by contradiction.

5. "If a is even then $\frac{(a+2)^2}{4}$ is even."

6. "If ab is even then a and b is even."

7. "If $a+b$ is odd then a is odd."

   Answer: If $a=2$ and $b=1$ then $a+b=3$ which is odd, yet a is not odd. $\square$

8. "If $a+b$ is even then a and b are both even."

9. "If $a+b$ is even then a and b are both odd."

   Answer: If $a=2$ and $b=2$ then $a+b$ is even but a and b are not odd. $\square$

10. "If a^2+a is even then a is even."

    Answer: If $a=1$ then a^2+a is even but a is not even. $\square$

11. "If a^2+a is even then a is odd."

12. "If a^2+b^2 is even then a and b are even."

13. "If the average of a and b is even then a or b is even."

    Answer: If $a=1$ and $b=3$ then the average is 2 which is even, yet neither one of a or b is even. $\square$

14. "If the average of a and b is odd then a or b is odd."

    Answer: If $a=2$ and $b=4$ then the average is 3 which is odd, yet neither one of a or b is odd. $\square$

15. "If the average of a and b is even then a or b is odd."

16. "If the average of a and b is odd then a or b is even."

17. "If a and b are integers then $(a+1)b$ or $(a+1)(b+1)$ is odd."

    Answer: If $a=1$ and $b=1$ then $(a+1)b=2$ and $(a+1)(b+1)=4$. Neither of these is odd. $\square$

2.2 Divisibility Proofs

Assume that a, b, c and d are integers and prove the following statements.

1. Every nonzero integer divides itself.

2. Every nonzero integer divides its negative.

   Proof: Let a be a nonzero number. We want to find an $r \in \mathbb{Z}$ so that $ar = -a$. Setting r to be the integer -1 completes the proof. $\square$

3. Every nonzero integer divides its square.

   Proof: Let a be any nonzero integer. We want to find an $r \in \mathbb{Z}$ so that $ar = a^2$. Simply set $r = a \in \mathbb{Z}$ to complete the proof. $\square$

4. Every nonzero integer divides zero.

5. One divides every integer.

6. Negative one divides every integer.

   Proof: Let a be an integer. We wish to find an intger k so that $(-1)k = a$. Set k to be the integer $-a$ and we are done. $\square$

7. An integer a divides $a^2 + a$.

8. If a is even then four divides $2a$.

9. If a is odd then four divides $(a + 1)^2$.

   Proof: Suppose a is odd. This means $a = 2r + 1$ for some integer r. Then $(a + 1)^2 = (2r + 2)^2 = (2(r+1))^2 = 2^2(r+1)^2 = 4(r+1)^2$. Set $s = (r+1)^2$ to get $4s = (a+1)^2$. As s is an integer, this shows four divides $(a + 1)^2$. $\square$

10. If a is odd then four divides $(a - 1)^2$.

11. If a is even then ten divides $5a$.

12. If a is even then ten divides $15a$.

    Proof: Suppose a is even and thus $a = 2r$ for some r in $\mathbb{Z}$. Then $15a = 15(2r) = 30r = 10(3r)$. Set s to be the integer $3r$. This shows that $15a = 10s$ thus proving ten divides $15a$. $\square$

13. If a is odd then eight does not divide a.

14. If a is odd then 2^{101} does not divide a.

    Proof: We show the contrapositive: If 2^{101} divides a then a is even. Suppose 2^{101} divides a . Then $2^{101}r = a$. Then $2(2^{100}r) = a$. Set $s = 2^{100}r$ to get $2s = a$. As s is an integer this proves a is even. $\square$

15. If a is odd then four divides $a-1$ or $a+1$.

    Proof: Let a be odd. Then $a = 2k+1$ for some k in the integers. We need to consider two cases.

    Case 1: (k is even)

    Here $k = 2l$ for some l in the integers. Then $a-1 = 2k+1-1 = 2k = 2(2l) = 4l$. Then $a-1 = 4l$ and as l is an integer, this shows $a-1$ is divisible by four.

    Case 2: (k is odd)

    Here $k = 2m+1$ for some integer m. Then $a+1 = 2k+1+1 = 2k+2 = 2(2m+1)+2 = 4m+4 = 4(m+1)$. Set n to be the integer $m+1$. Then $a+1 = 4n$ which shows $a+1$ is divisible by four. □

16. If a is odd then either $a+1$ or $a+3$ is divisible by four.

17. If a is odd then either $a+3$ or $a-3$ is divisible by four.

    Proof: Suppose a is odd, so $a = 2r+1$ for $r \in \mathbb{Z}$.

    Case 1: r is even. Here $r = 2s$ for $s \in \mathbb{Z}$. Thus $a = 2(2s)+1 = 4s+1$. Here $a+3 = 4s+1+3 = 4s+4 = 4(s+1)$. Setting $t = s+1$ shows that $4t = a+3$ and thus we have shown $a+3$ is divisible by 4.

    Case 2: r is odd. Here $r = 2s+1$ for $s \in \mathbb{Z}$. Thus $a = 2(2s+1)+1 = 4s+3$. Here $a-3 = 4s+3-3 = 4s$. Setting $t = s$ shows that $4t = a-3$ and thus we have shown $a-3$ is divisible by 4. □

18. If a is odd then a^2-1 is divisible by eight.

19. If a^2-1 is not divisible by four then a is even.

20. If $a+1$ or $a+2$ is divisible by three then a^2-1 is divisible by three.

21. If three does not divide a^2+2 then three does not divide $a-1$ and three does not divide $a-2$.

    Proof: We prove the contrapositive statement: If three divides $a-1$ or $a-2$ then three divides a^2+2. If three divides $a-1$ then $a-1 = 3k$ for $k \in \mathbb{Z}$. If three divides $a-2$ then $a-2 = 3k$ for $k \in \mathbb{Z}$. We thus split things into cases as either $a = 3k+1$ or $a = 3k+2$ for $k \in \mathbb{Z}$.

    Case 1: $a = 3k+1$. Here $a^2+2 = (3k+1)^2+2 = 9k^2+6k+1+2 = 9k^2+6k+3 = 3(3k^2+2k+1)$. Setting $m = 3k^2+2k+1 \in \mathbb{Z}$ we see that $a^2+2 = 3m$. This means that three divides a^2+2.

    Case 2: $a = 3k+2$. Note that $a^2+2 = (3k+2)^2+2 = 9k^2+12k+4+2 = 9k^2+12k+6 = 3(3k^2+4k+2)$. Set $m = 3k^2+4k+2 \in \mathbb{Z}$ to see $a^2+2 = 3m$ and conclude that three divides a^2+2. □

22. If a divides b then a divides $b+a$.

23. If a^2 divides b then a divides b.

    Proof: Assume that a^2 divides b. Thus $a^2k = b$ for some integer k. Then $a(ak) = b$. Set l to be the integer ak. Then we have $al = b$ and thus a divides b. □

24. If a^4 divides b then a^3 divides b.

25. If ab divides b^2 then a divides b.

26. If a divides b^3 then a divides b^4.

    Proof: Assume that a divides b^3. Thus $ak = b^3$ for some integer k. Then $b^4 = b^3b = akb$. Set l to be the integer kb. Then we have $b^4 = al$ showing a divides b^4. $\square$

27. If a divides b and b divides a then $a = b$ or $a = -b$.

    Proof: Suppose that $a|b$ and $b|a$. This means that $ar = b$ and $bs = a$ for some $r, s \in \mathbb{Z}$ and the definition of divides implies that $a \neq 0$. Plugging one equation into the other gives us $ars = a$ which means $ars - a = 0$ and $a(rs-1) = 0$. Since $a \neq 0$ we know that $rs - 1 = 0$ or that $rs = 1$. Since r and s are integers, this means they must both be 1 or both be -1. If they are both 1 then $a = b$ and if they are both -1 then $a = -b$. Either way, we are done. $\square$

28. If a divides b and a does not divide $b + c$ then a does not divide c.

    Proof: We will prove the contrapositive, that if a divides c then a doesn't divide b or a divides $b + c$. Recall $P \Rightarrow (Q \vee R) \equiv P \wedge \sim Q \Rightarrow R$ so this is equivalent to showing that if a divides c and a divides b then a divides $b + c$. This is proved earlier in these exercises, so the rest of the proof can be found above. $\square$

29. If a divides b and a does not divide $b - c$ then a does not divide c.

30. If $a + 1$ and $b + 1$ are divisible by three then $ab + 2$ is also divisible by three.

31. If $a + 1$ is divisible by three and $b + 2$ is divisible by three then $ab + 1$ is divisible by three.

32. If $a - 2$ and $b - 2$ are divisible by three then $ab - 1$ is also divisible by three.

33. If $a - 3$ and $b - 3$ are divisible by four then $ab - 1$ is also divisible by four.

34. If $a + 1$ and $b + 1$ are divisible by n then $ab - 1$ is also divisible by n.

    Proof: Assume that $a+1$ and $b+1$ are divisible by n which means $nr = (a+1)$ and $ns = (b+1)$ for some $r, s \in \mathbb{Z}$. Since $a = nr-1$ and $b = ns-1$ we know $ab-1 = (nr-1)(ns-1)-1 = n^2rs-nr-ns+1-1 = n(nrs-r-s)$. Set $t = (nrs-r-s) \in \mathbb{Z}$ to see that $nt = ab-1$ and therefore show n divides $ab-1$. $\square$

35. If a is divisible by three or b is divisible by 36 then ab is divisible by three.

    Proof: We proceed by cases[5].

    Case 1: $3|a$. Here $a = 3k$ for $k \in \mathbb{Z}$ and thus $ab = 3kb$. Set $l = kb \in \mathbb{Z}$ to see $ab = 3l$ thus showing ab is divisible by 3. $\square$

    Case 2: $36|b$. Here $b = 36k$ for $k \in \mathbb{Z}$ and thus $ab = 36ak = 3(12ak)$. Set $l = 12ak \in \mathbb{Z}$ to see $ab = 3l$ thus showing ab is divisible by 3. $\square$

36. If $a - 2$ and $b - 5$ are divisible by three then $ab + 2$ is also divisible by three.

    Proof: Suppose that $3|a-2$ and $3|b-5$. Then $3r = a-2$ and $3s = b-5$ for some r and s in $\mathbb{Z}$. This means $a = 3r+2$ and $b = 3s+5$. Now $ab+2 = (3r+2)(3s+5)+2 = 9rs+6s+15r+12 = 3(3rs+2s+5r+4)$. Set $t = 3rs + 2s + 5r + 4 \in \mathbb{Z}$ to get $ab + 2 = 3t$ and see that $ab + 2$ is divisible by 3. $\square$

37. If three divides a and four divides b then twelve divides ab.

[5] We cannot state "without loss of generality" here since a and b in our or statement have distinctly different properties.

38. If three divides a and six divides b then three divides $a+b$.

39. If a and b are odd then a^2+b^2+2 is divisible by four.

40. If a and b are odd then $ab+a+b+1$ is divisible by four.

    Proof: Assume that a and b are odd, so $a=2r+1$ and $b=2s+1$ for some $r,s\in\mathbb{Z}$. Now $ab+a+b+1=(2r+1)(2s+1)+(2r+1)+(2s+1)+1=4rs+4r+4s+4=4(rs+r+s+1)$. Set t to be the integer $rs+r+s+1$ to get $4t=ab+a+b+1$ and see that 4 divides $ab+a+b+1$. $\square$

41. If a and b are odd then a^2-1 and b^2-1 are both divisible by four.

42. If a and b are odd then $(a^2-1)(b^2-1)$ is divisible by sixteen.

43. If a and b are different parities then $(a^2-1)(b^2-1)$ is divisible by four.

    Proof: Without loss of generality assume that a is even and b is odd. Then $a=2r$ and $b=2s+1$ for some $r,s\in\mathbb{Z}$. Now $(a^2-1)(b^2-1)=((2r)^2-1)((2s+1)^2-1)=(4r^2-1)(4s^2+4s+1-1)=4(4r^2-1)(s^2+s)$. Setting t to be the integer $(4r^2-1)(s^2+s)$ shows that $4t=(a^2-1)(b^2-1)$ and thus 4 must divide $(a^2-1)(b^2-1)$. $\square$

44. If a divides b then a divides $b-ca$.

    Proof: If a divides b we know $ar=b$ for some integer r. Then $b-ca=ar-ca=a(r-c)$. We can set $s=r-c$, which is an integer, to see that $as=b-ca$ thus proving a divides $b-ca$. $\square$

45. If a divides b and c then a divides $b+c$.

    Proof: Suppose that a divides b and c. This means $ar=b$ and $as=c$ for some $r,s\in\mathbb{Z}$. Then $b+c=ar+as=a(r+s)$. Set $t=r+s\in\mathbb{Z}$ to see that $at=b+c$ and show a divides $b+c$. $\square$

46. If a divides b and c then a divides bc.

47. If a divides b or c then a divides bc.

    Proof: Without loss of generality[6], suppose a divides b. Then $ar=b$ for some $r\in\mathbb{Z}$. Thus $bc=arc$ and setting $s=rc\in\mathbb{Z}$ implies that a divides bc. $\square$

48. If a divides $b+c$ and a divides c then a divides b.

49. If a divides b and c then a divides $a+b+c$.

50. If a divides b and c then a divides $(b+c)^2$.

51. If a divides $b+c$ and $b-c$ then a divides c^2-b^2.

52. If a divides b then ac divides bc.

53. If a divides b and b divides c then a divides c.

    Proof: Suppose that $a|b$ and $b|c$. This means $ar=b$ and $bs=c$ for some $r,s\in\mathbb{Z}$. We must show that $a|c$ or that $at=c$ for some $t\in\mathbb{Z}$. As $c=bs=ars$, if we set $t=rs$, which is in $\mathbb{Z}$, then we see that $at=c$ completing the proof. $\square$

[6] We could also easily break this down into the two cases of $a|b$ and $a|c$ but as the equation is symmetric in b and c this allowed us to use this shortcut.

54. If ab divides c then a divides c and b divides c.

55. If ab divides bc then a divides c.

56. If a divides b and c then a^2 divides bc.

57. If a and b divide c then ab divides c^2.

    Proof: Assume that a and b divide c, which means $ar = c$ and $bs = c$ for some $r, s \in \mathbb{Z}$. Then $c^2 = arbs = ab(rs)$. Setting $t = rs \in \mathbb{Z}$ shows that $ab|c^2$. □

58. If a does not divide bc then a does not divide b.

59. If a does not divide bc then a does not divide b and a does not divide c.

60. If a divides $b - c$ and $c - d$, then a divides $b - d$.

    Proof: Suppose a divides $b - c$ and $c - d$. Thus $ar = b - c$ and $as = c - d$ for some integers r and s. Then $b - d = b - c + c - d = ar + as = a(r + s)$. Let t be the integer $r + s$. Then $b - d = at$ which shows a divides $b - d$.[7] □

61. If a divides $b + c$ and $c + d$, then a divides $b - d$.

62. If ab, bc and ac all divide d then $(abc)^2$ divides d^3.

    Proof: Assume that ab, bc and ac divide d, which implies $abr = bcs = act = d$ for some r, s, and t in $\mathbb{Z}$. Then $d^3 = (abr)(bcs)(act) = a^2b^2c^2rst = (abc)^2rst$. Let $u = rst \in \mathbb{Z}$. As $d^3 = (abc)^2u$ we see that $(abc)^2$ divides d^3. □

Assume that a, b, c and d are integers and disprove the following statements.

1. If a divides bc then a divides b and a divides c.

   Answer: If $a = 4, b = 8$ and $c = 2$ then a divides bc but a does not divide both b and c.

2. If a divides bc then a divides b or a divides c.

3. If a divides $b + c$ then a divides b and a divides c.

4. If a divides $b + c$ then a divides b or a divides c.

5. If a divides $b - c$ then a divides b and a divides c.

6. If a divides $b - c$ then a divides b or a divides c.

   Answer: If $a = 2, b = 5, c = 1$ then a divides $b - c$ but a does not divide b or c.

7. If a divides b and b divides a then $a = b$.

   Answer: If $a = 2$ and $b = -2$ then a divides b and b divides a but the two are not equal.

8. If 3 does not divide a then three divides $a - 1$.

   Answer: If $a = 2$ then three does not divide a or $a - 1$.

[7] Here we add zero by adding $-c + c$. We could also prove this by solving for both b and d in the two equations given. That method takes only a slight bit more work, but may require less thought.

9. If 3 does not divide a then three divides $a - 2$.

10. If four divides a then eight divides a.

11. If six divides a and 15 divides a then 6×15 divides a.

    Answer: If $a = 30$ then six divides a and fifteen divides a but $6 \times 15 = 90$ does not.

12. If $a + b$ and $b + c$ divide d, then $a + c$ divides d.

    Answer: When $a = 2, b = 1, b = 2$ and $d = 6$ then $a + b$ and $b + c$ both divide d, but $a + c$ does not divide d.

13. If a is odd then a does not divide 2^{101}.

14. If ab divides cd then a divides c and b divides d.

15. If ab divides cd then a divides c or d.

16. If a divides b and c divides d then $a + b$ divides $c + d$.

    Answer: If $a = 2, b = 4, c = 3$, and $d = 6$ then a divides b and c divides d, but $a + b = 6$ does not divide $c + d = 9$.

17. If a divides b and c divides d then $a - b$ divides $c - d$.

2.3 Modular Arithmetic Proofs

Prove the following statements. Assume that a, b, c, d, and n are all integers and that $n \geq 2$

1. $0 \equiv n \pmod{n}$.
2. $-n \equiv n \pmod{n}$.

   Proof: We must show n divides $n - -n$. That is, we must find a k so $nk = n + n$. Set k to be the integer two. This completes the proof. $\square$
3. $a \equiv a \pmod{n}$.
4. $a \equiv n + a \pmod{n}$.
5. $a \equiv bn + a \pmod{n}$.

   Proof: We must show n divides $bn + a - a$. This means that we must find a k so $nk = bn$. Setting k to be the integer b completes the proof. $\square$.
6. If $a \equiv b \pmod{n}$ then $b \equiv a \pmod{n}$.

   Proof: Assume $a \equiv b \pmod{n}$. Thus n divides $b - a$ and nr equals $b - a$ for some $r \in \mathbb{Z}$. Now $a - b = -(b - a) = -nr = n(-r)$. If we set $s = -r$ which is also an integer, then we see that n divides $a - b$ and thus $b \equiv a \pmod{n}$.
7. If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ then $a \equiv c$ (mod n.)

   Proof: Assume that $n|b - a$ and $n|c - b$ so there are $r, s \in \mathbb{Z}$ so $nr = b - a$ and $ns = c - b$. Then $c - a = c - b + b - a = ns + nr = n(s + r)$. Set $t = s + r \in \mathbb{Z}$. to see $n|c - a$ and thus $a \equiv c$ (mod n.) $\square$
8. If $a \equiv b \pmod{n}$ then $a + c \equiv b + c \pmod{n}$.
9. If $a \equiv b \pmod{n}$ then $a - c \equiv b - c \pmod{n}$.
10. If $a \equiv b \pmod{n}$ then $ac \equiv bc \pmod{n}$.
11. If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$ then $a + c \equiv b + d$ (mod n.)

    Proof: Assume that $n|b - a$ and $n|d - c$ so there are $r, s \in \mathbb{Z}$ so $nr = b - a$ and $ns = d - c$. Then $b + d - (a + c) = b + d - a - c = b - a + d - c = nr + ns = n(r + s)$. Set $t = r + s \in \mathbb{Z}$. to see $n|b + d - (a + c)$ and thus $a + c \equiv b + d$ (mod n.) $\square$
12. If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$ then $a - c \equiv b - d$ (mod n.)

    Proof: Assume $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, thus n divides both $b - a$ and $d - c$. This means $nr = b - a$ and $ns = d - c$ for some $r, s \in \mathbb{Z}$. Then $b - d - (a - c) = b - a - d + c = b - a - (d - c) = nr - ns = n(r - s)$. Set $t = r - s \in \mathbb{Z}$ to see $b - d - (a - c) = nt$, thus showing that n divides $b - d - (a - c)$ and $a - c \equiv b - d$ (mod n.) $\square$
13. If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$ then $ac \equiv bd$ (mod n.)

    Proof: Assume that $n|b - a$ and $n|d - c$ so there are $r, s \in \mathbb{Z}$ so $nr = b - a$ and $ns = d - c$. Then $bd - ac = bd - bc + bc - ac = b(d - c) + (b - a)c = bns + nrc = n(bs + rc)$. Set $t = bs + rc \in \mathbb{Z}$. to see $n|bd - ac$ and thus $ac \equiv bd$ (mod n.) $\square$

14. If $a \equiv 3 \pmod 4$ and $a \equiv 3 \pmod 6$ then a^2 is divisible by 3.

15. If $a \equiv 3 \pmod 4$ and $a \equiv 3 \pmod 6$ then $a^2 \equiv 3$ (mod 6.)

    Proof: Assume that $a \equiv 3 \pmod 4$ and $a \equiv 3 \pmod 6$, meaning $4|a-3$ and $6|a-3$. Thus $4k = a-3$ and $6l = a-3$ for some $k, l \in \mathbb{Z}$. Then $a^2 = a \cdot a = (4k+3)(6l+3) = 24kl+18l+12k+9 = 6(4kl+3l+2k+1)+3$. Setting $m = 4kl + 3l + 2k + 1 \in \mathbb{Z}$ shows us $a^2 = 6m + 3$ or that $6|a^2 - 3$. This means $a^2 \equiv 3$ (mod 6.)

16. If $a \equiv 2 \pmod 4$ and $b \equiv 2 \pmod 4$ then ab is divisible by 4.

17. If $a \equiv 3 \pmod 4$ and $b \equiv 1 \pmod 4$ then $ab + b$ is divisible by 4.

18. If a is odd then $a^2 \equiv 1 \pmod 4$.

    Proof: Assume a is odd and thus $a = 2k+1$ for $k \in \mathbb{Z}$. Then $1-a^2 = 1-(2k+1)^2 = 1-(4k^2-4k+1) = -4k^2 + 4k = 4(-k^2 + k)$. Setting l to be the integer $-k^2 + k$ we see that four divides $1 - a^2$ and thus $a^2 \equiv 1 \pmod 4$. □

19. If $1 \equiv a \pmod 5$ or $4 \equiv a \pmod 5$ then $1 \equiv a^2 \pmod 5$.

20. If $2 \equiv a \pmod 5$ or $3 \equiv a \pmod 5$ then $-1 \equiv a^2 \pmod 5$.

    Proof: We break things down into two cases.

    Case 1: $a \equiv 2 \pmod 5$

    Here 5 divides $a - 2$ so $5k = a - 2$ for some integer k. Then $a^2 - -1 = a^2 + 1 = (2 + 5k)^2 + 1 = 4 + 20k + 25k^2 + 1 = 5 + 20k + 25k^2 = 5(1 + 4k + 5k^2)$. Set $l = 1 + 4k + 5k^2 \in \mathbb{Z}$ to see that $5|a^2 + 1$ which shows $-1 \equiv a^2 \pmod 5$.

    Case 2: $a \equiv 3 \pmod 5$

    We know $5m = a-3$ for some integer m. Then $a^2 - -1 = a^2+1 = (5m+3)^2+1 = 25m^2+30m+9+1 = 25m^2 + 30m + 10 = 5(5m^2 + 6m + 2)$. Set $n = 5m^2 + 6m + 2$, which is an integer, to see that $n|a^2 + 1$, showing $-1 \equiv a^2 \pmod 5$. □

21. If a is even then $a^3 \equiv 0 \pmod 8$.

22. If a is even then $(a - 1)(a^2 + a + 1) \equiv 7 \pmod 8$.

23. If a is odd then $1 \equiv a^4 \pmod 8$.

    Proof: Assume a is odd and thus $a = 2r + 1$ for $r \in \mathbb{Z}$. Then $a^4 - 1 = 16k^4 + 32k^3 + 24k^2 + 8k + 1 - 1 = 8(2k^4 + 4k^3 + 3k^2 + k)$. This shows $8|a^4 - 1$ which means $1 \equiv a^4 \pmod 8$. □

24. If $a \equiv 1 \pmod 3$ then $a^3 \equiv 1 \pmod 9$.

25. If $a \equiv 2 \pmod 3$ then $a^3 \equiv -1 \pmod 9$.

2.4 Proofs Involving Rational and Irrational Numbers

Prove the following statements. Feel free to use the fact that the integers are closed under addition and multiplication. Also feel free to use the fact that a product of non-zero numbers is also non-zero and that a rational number is zero iff the numerator is zero.

1. Zero is a rational number.

   Proof: As zero is equal to $\frac{a}{b}$ for $a = 0$ and $b = 1(\neq 0)$ we know zero is rational.

2. Negative one is a rational number.

3. The square of a rational number is rational.

   Proof: Suppose a is a rational number. Then $a = \frac{b}{c}$ for some b and c in the integers with $c \neq 0$. Then $a^2 = (\frac{b}{c})^2 = \frac{b^2}{c^2}$. Since $c \neq 0$ we know $c^2 \neq 0$. As both b^2 and c^2 are integers, this shows a^2 is rational. $\square$

4. The reciprocal of a non-zero rational number is rational.

5. Half of a rational number is a rational number.

6. The sum of a rational number with itself is rational.

   Proof: Suppose $\frac{a}{b}$ is a rational number. Thus a and b are integers with $b \neq 0$. Then $\frac{a}{b} + \frac{a}{b} = \frac{2a}{b}$. As $2a$ and b are integers with $b \neq 0$, we know this is rational. $\square$

7. The sum of a rational with an integer is rational.

8. The quotient of a rational with a non-zero integer is rational.

9. The product of two rational numbers is rational.

10. The quotient of a rational with a non-zero rational is rational.

    Proof: Suppose $\frac{a}{b}$ and $\frac{c}{d}$ are rationals with the latter being non-zero. That means $c \neq 0$. We also know that a, b, c, and d are integers, and that b and d are non-zero. Then $(\frac{a}{b})/(\frac{c}{d}) = \frac{ad}{bc}$. As b and c are both non-zero, so is bc. Since we also know ad and bc are integers, this is a rational number. $\square$

11. The sum of two rational numbers is rational.

    Proof: Suppose that a and b are rational numbers. Then $a = p/q$ and $b = r/s$ for $p, q, r, s \in \mathbb{Z}$, $q \neq 0$ and $s \neq 0$. Notice $a + b = p/q + r/s = ps/qs + rq/sq = (ps + rq)/qs$. Set $u = ps + rq$ and $v = qs$. These are in $\mathbb{Z}$ due to closure, and since q and s are nonzero, so is v. This shows $a + b = u/v$ for $u, v \in \mathbb{Z}$ and $v \neq 0$. $\square$

12. The difference of two rational numbers is rational.

13. The average of two integers is a rational number.

14. The average of two rational numbers is a rational number.

15. The product of a rational and an integer is rational.

    Proof: Suppose that a is rational and b is an integer. We can then write $a = c/d$ for $c, d \in \mathbb{Z}$ with $d \neq 0$. Then $ab = (c/d)b = (cb)/d$. We know cb and d are in $\mathbb{Z}$ and $d \neq 0$ so this shows ab is rational. $\square$

16. If three times a number is irrational then the number is irrational.

17. If the sum of one and a number is irrational then the number is irrational.

    Proof: We take the contrapositive and instead show that is a number is rational then one plus that number is rational. Call our rational number a and write it as $a = \frac{r}{s}$ for $r, s \in \mathbb{Z}$ with $s \neq 0$. Then $a + 1 = \frac{r}{s} + \frac{s}{s} = \frac{r+s}{s}$. Since both $r + s$ and s are integers with $s \neq 0$ we get that $a + 1 \in \mathbb{Q}$. $\square$

18. If the square of a number is irrational then the number is irrational.

19. If the cube of a number is irrational then the number is irrational.

20. If $a + b$ is rational and a is rational then b is rational.

21. If $a + b$ is rational and a is irrational then b is irrational.

    Proof: We take the contrapositive to get the statement: If b is rational then $a + b$ is irrational or a is rational. As $P \Rightarrow (Q \vee R) \equiv (P \wedge \sim Q) \Rightarrow R$ this statement is equivalent to: If b is rational and $a + b$ is rational then a is rational. This is the statement we choose to show.

    Suppose b and $a + b$ are rational. Then $b = \frac{c}{d}$ and $a + b = \frac{e}{f}$ for $c, d, e, f \in \mathbb{Z}$ and d and f not equal to zero. Then $a = (a + b) - b = \frac{e}{f} - \frac{c}{d} = \frac{ed-cf}{fd}$. As $ed - cf$ and fd are integers with $fd \neq 0$, this shows a is rational. $\square$

22. If $a + b$ is irrational and a is rational then b is irrational.

23. If ab is rational and a is a non-zero rational then b is rational.

    Proof: Assume that ab is rational and a is a non-zero rational. Thus $ab = \frac{c}{d}$ and $a = \frac{e}{f}$ with $c, d, e, f \in \mathbb{Z}$ and non-zero d, e and f. Then $\frac{e}{f} \cdot b = \frac{c}{d}$ so $b = \frac{cf}{de}$. We know de is not zero since d and e are not zero. As cf and de are integers, this shows b is rational. $\square$

24. If ab is rational and a is irrational and b is nonzero, then b is irrational.

25. If ab is irrational and a is rational then b is irrational.

26. If the average of two numbers is irrational they are not both rational.

    Proof: We take the contrapositive and instead show that if both numbers are rational then their average is rational[8]. Suppose that a and b are rational numbers. Then $a = \frac{p}{q}$ and $b = \frac{r}{s}$ for $p, q, r, s \in \mathbb{Z}$, $q \neq 0$ and $s \neq 0$. Notice $\frac{a+b}{2} = \frac{1}{2}(a + b) = \frac{1}{2}(\frac{p}{q} + \frac{r}{s}) = \frac{1}{2}(\frac{ps}{qs} + \frac{rq}{sq}) = \frac{1}{2}(\frac{ps+rq}{qs}) = \frac{ps+rq}{2qs}$. Set $u = ps + rq$ and $v = 2qs$. These are in $\mathbb{Z}$ due to closure, and since q and s and 2 are nonzero, so is v. This shows $\frac{a+b}{2} = \frac{u}{v}$ for $u, v \in \mathbb{Z}$ and $v \neq 0$ showing the average is rational. $\square$

27. If the average of two numbers is rational, then they are both rational or both irrational.

[8] When negating the second part we had to realize that if it's not the case that at least one number is irrational then neither is irrational, so both are rational.

28. If the product of three numbers is irrational then at least one is irrational.

Disprove the following statements by finding a counterexample.

1. The sum of a rational and an integer is an integer.
2. The product of a rational and an integer is an integer.
   Counterexample: The product of the rational $\frac{1}{2}$ and the integer 1 is $\frac{1}{2}$ which is not an integer. □
3. The quotient of two integers is rational.
   Counterexample: If $a = 1$ and $b = 0$ then the quotient is not rational[9].
4. The quotient of a rational and an integer is rational.
5. The quotient of two rational numbers is rational.
6. The square root of a rational number is rational.
   Counterexample: $r = 2$ is rational but $\sqrt{r}$ is not. □
7. The reciprocal of a rational number is rational.
8. The sum of two irrational numbers is irrational.
9. The difference of two irrational numbers is irrational.
10. The product of two irrational numbers is irrational.
11. The quotient of two irrational numbers is irrational.
12. The square of an irrational number is irrational.
    Counterexample: $\sqrt{2}$ is irrational but its square is two, which is not. □
13. The cube of an irrational number is irrational.
14. The average of two irrational numbers is irrational.
    Counterexample: The average of $\sqrt{2}$ and $6 - \sqrt{2}$ is 3 which is rational. □
15. The product of a rational number and an irrational number is irrational.
16. The product of a rational number and an irrational number is rational.

[9] In fact, it isn't even a number.

2.5 Positivity Proofs

If a real number x is positive, we write $0 < x$. We say $a < b$ if $b - a$ is positive (and write $0 < b - a$.)

We assume the following axioms:

- One is a positive number.
- A sum or product of positive numbers is positive. [Closure]
- Every number is exclusively either positive, negative or zero. [Tricotomy Principle]

Prove the following statements over $\mathbb{R}$. You may assume the standard laws of algebra (multiplication and addition are commutative, associative, the distributive law holds, etc.) Recall that x^n is the product $x \times x \times \cdots x$ of x times itself n times.

1. The number $1 + 1$ is positive [10].
2. The square of a positive real number is positive.
3. If cube of a positive number is positive.

   Proof: Assume x is positive. Then x^2 is a product of two positive numbers, so it is positive. Since both x and x^2 are positive, their product, which is x^3, must also be positive. □
4. The product of any three positive number is positive.

   Proof: Let x, y and z be positive numbers. Then xy is positive, so (xy) times z is a product of two positive numbers, and thus is positive. □
5. If x is positive then x^4 is positive.
6. If x is positive then $0 < x$.

   Proof: As x is positive, so is $x + 0$. As $0 = -0$ we know $x - 0$ is positive, which means $0 < x$. □
7. If $0 < x$ then x is positive.

   Proof: Assume $0 < x$, which means $x - 0$ is positive. As $x - 0 = x$, we know x is positive. □
8. If $1 < x$ then x is positive.

   Proof: If $1 < x$ then $x - 1$ is positive. As one is positive, the sum $x - 1 + 1 = x$ is also positive. □
9. If x is positive then $x^2 + x$ is positive.
10. If $x - 1$ is positive then $x^2 - x$ is positive.

    Proof: Assume $x - 1 > 0$. This means $x - 1$ plus 1 must be positive since it is a sum of positive numbers and thus x is positive. Now $x^2 - x = x(x - 1)$ is a product of two positive numbers and hence is positive. □
11. If $x - 1$ is positive then so is $x^2 - 1$.

[10] The name we give to this number is two.

12. If x is positive then so is $x^2 + x + 1$.

13. If $x < y$ and x is positive, then y is positive.

    Proof: We know $y - x$ and x are both positive, so their sum y is also positive. $\square$

14. If $x < y$ then $-y < -x$.

    Proof: Assume $y - x$ is positive. As $y - x = (-x) - (-y)$ we get that $-y < -x$. $\square$

15. If $-y < -x$ then $x < y$.

    Proof: Assume $-y < -x$ so $-x - (-y)$ is positive. As this equals $y - x$, we know $y - x$ is positive and $x < y$. $\square$

16. If $x < y$ then $x + z < y + z$.

    Proof: Assume $x < y$, so $y - x$ is positive. As $y - x = y + z - x - z = (y + z) - (x + z)$ we know $(y + z) - (x + z)$ is positive, which shows us $x + z < y + z$. $\square$

17. If $x < y$ then $x - z < y - z$.

18. If $x < y$ and z is positive, then $xz < yz$.

    Proof: Assume $y - x$ and z are positive. Then their product $(y - x)z = (yz) - (xz)$ is positive. This tells us $xz < yz$. $\square$

19. If $w < x$ and $y < z$ then $w + y < x + z$.

    Proof: We know $x - w$ and $z - y$ are positive. Thus their sum is positive. Since $(x - w) + (z - y) = (x + z) - (w + y)$ we get that $w + y < x + z$. $\square$

20. If $x < y$ and $y < z$ then $x < z$.

    Proof: Assume $x < y$ and $y < z$. Thus $0 < y - x$ and $0 < z - y$, which tells us that both $y - x$ and $z - y$ are positive. Since a sum of positive numbers is positive, we know $y - x + z - y = z - x$ is positive which tells us $x < z$. $\square$

21. If x and y are positive numbers and $x < y$ then $y^2 - x^2$ is positive.

22. If x and y are positive numbers and $x < y$ then $xy^2 - yx^2$ is positive.

23. If $w < x$ and $y < z$ and all four are positive, then $wy < xz$.

    Proof: We know $x - w$ and $z - y$ are positive so $(x - w)(z - y) = xz - wz - xy + wy$ is positive. As wz and xy are products of two positive numbers, they are both positive, as well as their sum $wz + xy$. We now know that $xz - wz - xy + wy + (wz + xy) = xz - wy$ is positive and hence $wy < xz$. $\square$

24. If $w < x$ and $y < z$ and both z and w are positive, then $wy < xz$.

    Note: This is very similar to the last problem, but we are asked to prove the same thing while assuming less.

    Proof: We know $(x-w)$ and z are positive, so $z(x-w)$ is positive. We know $(z-y)$ and w are positive, so $w(z-y)$ is positive. Putting these together we get that $z(x-w)+w(z-y) = zx-zw+wz-yw = zx-yw$ is positive. This shows us $wy < xz$. $\square$

25. Prove for any x and y that at most one of the following holds: $a < b$, $b < a$ or $a = b$.

    Proof: Start by assuming $a = b$. Here $0 = b - a$ and $0 = a - b$. If $a < b$ then $b - a$ is positive, which means that zero is positive. If $b < a$ then $a - b$ is positive, which means that zero is positive. Either way, we get a contradiction. Thus if $a = b$ then neither of the other two statements hold.

    Now we only need show that $a < b$ and $b < a$ both cannot happen. Suppose $a < b$ and $b < a$. Then $b - a$ and $a - b$ are both positive. This means their sum $b - a + a - b = 0$ must also be positive, which is a contradiction. $\square$

26. If x is either positive or negative then $-x$ is either positive or negative. [11]

    Proof: Suppose x is non-zero, but $-x$ is zero. Then $x = -(-x) = -0 = 0$, a contradiction. $\square$

27. If x is positive then $-x$ is negative.

    Proof: If x is positive then x cannot equal zero by the Law of Trichotomy. Since x is non-zero, by problem 26, we know $-x$ is either positive or negative. If $-x$ is positive then $x + (-x)$ is a sum of two positive numbers, hence positive. Thus zero is a positive number, which is a contradiction. We conclude that $-x$ can only be negative. $\square$

28. If x is negative then $-x$ is positive.

    Proof: We take the contrapositive and use the Law of Tricotomy to rewrite our statement. We get: "If $-x$ is positive or zero then x is negative or zero." If $-x$ is zero then $x = -(-x) = -0 = 0$, so we are done in that case. This leaves the case where $-x$ is positive.

    If $-x$ is positive then by problem 27, we can conclude $-(-x)$ is negative, which means x is negative. $\square$

29. If x is negative and y is positive then $x < y$.

    Proof: By problem 28 we know $-x$ is positive. Thus $y + (-x) = y - x$ is positive, which gives us $x < y$. $\square$

30. If $x < 0$ then x is negative.

    Proof: Assume $0 - x$ is positive. As this equals $-x$, we know $-x$ is positive and by problem 27, we get that $-(-x) = x$ is negative. $\square$

31. If x is negative then $x < 0$.

    Proof: Assume x is negative. Then $-x$ is positive so $0 + -x = 0 - x$ is positive, which shows us $x < 0$. $\square$

32. A sum of negative numbers is negative.

    Proof: Suppose x and y are negative. Then $-x$ and $-y$ are positive. Thus $-x - y = -(x + y)$ is positive. By problem 27, we get that $-(-(x + y)) = x + y$ is negative. $\square$

33. A product of negative numbers is positive.

    Proof: Suppose x and y are negative. By problem 28, we know $-x$ and $-y$ are positive. Then $(-x)(-y) = xy$ is a sum of positive numbers, which completes our proof. $\square$

[11] We could rewrite this as: "If x is non-zero then so is $-x$."

34. The product of a positive and negative number is negative.

    Proof: Suppose x is positive and y is negative. Then by problem 28, we know $(-y)$ is positive. This tells us $x(-y) = -(xy)$ is positive. By problem 27, we know xy is negative. □

35. The cube of a negative number is negative.

36. If x is negative then x^4 is positive.

37. If $x < y$ and z is negative, then $yz < xz$.

    Proof: Since $y - x$ and $-z$ are positive we know $(y - x)(-z) = -yz + xz = xz - yz$ is positive. Thus $yz < xz$. □

38. If $x < y$ and y is negative, then x is negative.

    Proof: We know $y - x$ is positive and by problem 28, $-y$ is positive. Thus $(y - x) + (-y) = -x$ is a sum of positive numbers, hence positive. Since $-x$ is positive, by problem 27, we conclude that x is negative. □

39. If x is positive and $x + y$ is negative then y is negative.

    Proof: From problem 28, we know $-(x + y)$ is positive. As x is positive, the sum $x + -(x + y) = -y$ is positive. By problem 27 we know y is negative. □

40. If x is positive then $1/x$ is positive.

    Proof: If $1/x$ is negative then $x(1/x) = 1$ is a product of a positive and negative, hence negative. This is a contradiction. If $1/x = 0$ then $(x^2)(1/x) = x^2(0)$, which shows $x = 0$, contradicting the fact that x is positive. This leaves one case left, that where $1/x$ is positive. □

41. If $x < y$ and z is positive, then $x/z < y/z$.

    Proof: By problem 40, we know $1/z$ is positive. Thus $(y - x)(1/z) = y/z - x/z$ is positive, hence $x/z < y/z$. □

42. If x is negative, then $1/x$ is negative.

    Proof: We know $-(1/x)$ is positive from problems 28 and 40. Thus by problem 27 $-(-(1/x)) = 1/x$ is positive. □

43. If $1 < x$ then $1/x < 1$.

    Proof: If $1 < x$ then from problem 8, we know x is positive. By problem 40, we know $1/x$ is positive. Since $x - 1$ and $1/x$ are positive, we get that $(x - 1)(1/x) = 1 - 1/x$ is positive and hence $1/x < 1$. □

44. If x and y are either both positive or both negative, and $x < y$ then $1/y < 1/x$.

    Proof: We know $(1/x)(1/y)$ is positive by problem 40. We also know $y - x$ is positive. Thus $(1/x)(1/y)(y - x) = 1/x - 1/y$ is also positive, which tells us $1/y < 1/x$. □

45. Suppose exactly one of x and y is positive, and the other is negative. If $x < y$ then $1/x < 1/y$.

2.6 Proof by Contradiction

Use proof by contradiction to prove the following statements about parity. Recall that one is not an even number.

1. If the product ab is odd, then both a and b are odd.

   Proof: Assume not. Then at least one of them is even. Without loss of generality we can assume a is even, so we know $a = 2k$ for some integer k. Then $ab = 2kb$. Setting l to be the integer kb shows us that ab is even, a contradiction. □

2. An integer cannot be both even and odd.

   Proof: Suppose that a is both even and odd. Thus $a = 2k$ and $a = 2l + 1$ for some integers k and l. Then we know $2k = 2l + 1$ so $1 = 2k - 2l = 2(k - l)$. Setting m to be the integer $k - l$ shows us that one is an even number, which is a contradiction. □

3. A Pythagorean triple cannot have exactly one odd number[12].

   Proof: Assume that such a triple exists.

   Case 1: c is odd. Here $a^2 + b^2 = c^2$ where $a = 2r$, $b = 2s$ and $c = 2t + 1$ for some $r, s, t \in \mathbb{Z}$. Then $4r^2 + 4s^2 = 4t^2 + 4t + 1$ so $1 = 4r^2 + 4s^2 - 4t^2 - 4t = 2(2r^2 + 2s^2 - 2t^2 - 2t)$. Setting u to be the integer $2r^2 + 2s^2 - 2t^2 - 2t$ shows that one is an even number, a contradiction.

   Case 2: a or b is odd. Without loss of generality, assume that it is a. Here $a^2 + b^2 = c^2$ where $a = 2r + 1$, $b = 2s$ and $c = 2t$ for some $r, s, t \in \mathbb{Z}$. Then $4r^2 + 4r + 1 + 4s^2 = 4t^2$ so $1 = 4r^2 + 4r + 4s^2 + 4t^2 = 2(2r^2 + 2r + 2s^2 + 2t^2)$. Setting u to be the integer $2r^2 + 2r + 2s^2 + 2t^2$ shows that one is even, giving us a contradiction. □

4. A Pythagorean triple cannot have three odd numbers.

5. In a Pythagorean triple, either a or b must be even.

   Proof: Suppose that a and b are both odd. Then $a = 2r + 1$ and $b = 2s + 1$ for some $r, s \in \mathbb{Z}$. so $a^2 + b^2 = 4r^2 + 4r + 1 + 4s^2 + 4s + 1 = 4(r^2 + r + s^2 + s) + 2$.

   Case 1: c is odd. Here $c = 2t + 1$ so $c^2 = 4t^2 + 4t + 1$. As a, b, c is a Pythagorean triple, we know that $4(r^2 + r + s^2 + s) + 2 = 4t^2 + 4t + 1$, showing $1 = 4(r^2 + r + s^2 + s) + 2 - 4t^2 - 4t = 2(2r^2 + 2r + 2s^2 + 2s + 1 - 2t^2 - 2t)$. Since $2r^2 + 2r + 2s^2 + 2s + 1 - 2t^2 - 2t$ is an integer, this shows 1 is an even number, a contradiction. □

   Case 2: c is even. Here $c = 2t$ thus $c^2 = 4t^2$. We get that $4t^2 = 4(r^2 + r + s^2 + s) + 2$ so $2t^2 = 2(r^2 + r + s^2 + s) + 1$. Here $1 = 2(t^2 - r^2 - r - s^2 - s)$. As $t^2 - r^2 - r - s^2 - s$ is an integer, this shows one is even, giving us a contradiction. □

Use proof by contradiction to prove the following statements about positivity and inequality. Assume all the rules and results from the positivity handout. Consider that a least element a is one where $b < a$ cannot happen for any b and a greatest element c is one where $c < d$ cannot happen for any d.

1. The numbers x and $-x$ cannot both be positive.

[12] A Pythagorean triple is a collection of three integers, a, b and c so that $a^2 + b^2 = c^2$.

2. It is impossible to have both $x < y$ and $y < x$ for any real numbers x and y.

   Proof: Suppose not. Then $y - x$ and $x - y$ are both positive. Thus their sum $y - x + x - y = 0$ is positive. This contradicts the Law of Tricotomy. $\square$

3. The integers have no greatest element.

   Proof: Suppose a is the greatest integer. Then $a < a + 1$ because $(a + 1) - a$ is one, which is positive. This contradicts that a is the greatest element. $\square$

4. The integers have no least element.

5. There is no least element in the set $\{\frac{1}{n} : n \in \mathbb{N}\}$. You may assume all the elements of $\mathbb{N}$ are positive.

   Proof: Suppose not. Then for some $k \in \mathbb{N}$, $\frac{1}{k}$ is the least element. $\frac{1}{k+1} < \frac{1}{k}$ because $\frac{1}{k} - \frac{1}{k+1} = \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} = \frac{k+1}{k(k+1)}$ is a product and quotient of positive numbers, hence positive. This is a contradiction. $\square$

Use a proof by contradiction to prove the following statements about rational and irrational numbers. Feel free to assume $\sqrt{2}$ is irrational.

1. The rational number $\frac{1}{2}$ is not an integer.

   Proof: Suppose it is. Then $\frac{1}{2} = k$ for some $k \in \mathbb{Z}$. Thus $1 = 2k$, and since k is an integer, this shows 1 to be an even number. This gives us a contradiction. $\square$

2. The number $\sqrt{8}$ is irrational.

   Proof: Suppose not. Then $2\sqrt{2} = \sqrt{8} = \frac{a}{b}$ for some integers a and b with b not equal to zero. Thus $\sqrt{2} = \frac{a}{2b}$. As both 2 and b are non-zero, so is $2b$. We also know a and $2b$ are integers which shows $\sqrt{2}$ is a rational number. This is a contradiction. $\square$

3. The number $2 - \sqrt{2}$ is irrational.

4. The product of a non-zero integer and $\sqrt{2}$ is irrational.

5. A non-zero integer divided by $\sqrt{2}$ is irrational.

6. The number $\sqrt{2}$ divided by a non-zero integer is irrational.

7. The numbers $\sqrt{6}$ and $\sqrt{3}$ can not both be rational.

   Proof: Suppose they both are rational. Thus $\sqrt{6} = \frac{a}{b}$ and $\sqrt{3} = \frac{c}{d}$ for $a, b, c, d \in \mathbb{Z}$, $b \neq 0$ and $d \neq 0$. Then their product $\sqrt{6}\sqrt{3}$ is $\frac{ac}{bd}$. This product also equals $\sqrt{18}$ or $3\sqrt{2}$. Since $3\sqrt{2} = \frac{ac}{bd}$ we know $\sqrt{2} = \frac{ac}{3bd}$. Since $ac, 3bd \in \mathbb{Z}$ and $3bd \neq 0$, this shows $\sqrt{2}$ is a rational number, a contradiction. $\square$

8. The number $\sqrt{1 + \sqrt{2}}$ is irrational.

9. The number $\sqrt{1 + \sqrt{1 + \sqrt{2}}}$ is irrational.

   Proof: Suppose not. Then for some integers a and b with $b \neq 0$, we get $\frac{a}{b} = \sqrt{1 + \sqrt{1 + \sqrt{2}}}$. Thus $\sqrt{1 + \sqrt{2}} = (\frac{a}{b})^2 - 1 = \frac{a^2}{b^2} - 1 = \frac{a^2}{b^2} - \frac{b^2}{b^2} = \frac{a^2 - b^2}{b^2}$. This implies $\sqrt{2} = (\frac{a^2 - b^2}{b^2})^2 - 1 = (\frac{a^4 - 2a^2b^2 + b^4}{b^2}) - \frac{b^2}{b^2} = \frac{a^4 - 2a^2b^2 + b^4 - b^2}{b^2}$. As both $a^4 - 2a^2b^2 + b^4 - b^2$ and b^2 are integers and b^2 is not zero, we get that $\sqrt{2}$ is a rational number, which is a contradiction. $\square$

10. The number $\sqrt{2}^3$ is irrational.

    Proof: Assume that $\sqrt{2}^3$ is rational. Thus is equals $\frac{a}{b}$ for $a, b \in \mathbb{Z}$ and $b \neq 0$. Then $\sqrt{2}^3 = \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} = 2\sqrt{2} = \frac{a}{b}$ so $\sqrt{2} = \frac{a}{2b}$. Since $a, 2b \in \mathbb{Z}$ and $2b \neq 0$ we have shown $\sqrt{2}$ is rational, a contradiction. $\square$

11. Prove that $\sqrt[3]{2}$ is irrational.

12. Twice an irrational number is an irrational number.

    Proof: Suppose not. Then there is some irrational number x where $2x$ is not irrational. Thus we can write $2x = \frac{b}{c}$ where $b, c \in \mathbb{Z}$ and $c \neq 0$. This means $x = \frac{b}{2c}$. As $2c$ is a nonzero integer and b is an integer, we have shown x to be rational, a contradiction. $\square$

13. Half of an irrational number is an irrational number.

14. The square root of an irrational number is irrational.

    Proof: Suppose there is some irrational number x where $\sqrt{x}$ is rational. Then $\sqrt{x} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ and $b \neq 0$. This means $x = \frac{a}{b}^2 = \frac{a^2}{b^2}$. As a^2 and b^2 are integers with b^2 non-zero, we have shown x to be rational. This gives us a contradiction. $\square$

15. The sum of an integer and an irrational number is irrational.

    Proof: Suppose the sum of the integer a with the irrational number x is not irrational. Then $a + x = \frac{b}{c}$ for some $b, c \in \mathbb{Z}$ and $c \neq 0$. Then $x = \frac{b}{c} - a = \frac{b}{c} - \frac{ac}{c} = \frac{b-ac}{c}$. As $b - ac$ and c are integers with $c \neq 0$, this shows that x is a rational number, which is a contradiction. $\square$

16. The sum of a rational number and an irrational number is irrational.

17. The product of a non-zero integer and an irrational number is irrational.

18. The product of a non-zero rational number and an irrational number is irrational.

19. An irrational number divided by a non-zero rational number is irrational.

20. A non-zero rational number divided by an irrational number is irrational.

    Proof: Suppose not. Then the product of some non-zero rational $\frac{a}{b}$ divided by an irrational number x gives us a rational number $\frac{c}{d}$. Here $a, b, c, d \in \mathbb{Z}$ with $b \neq 0$ and $d \neq 0$. As $(\frac{a}{b})/x = \frac{c}{d}$ we know $\frac{a}{b} = \frac{c}{d}x$. Now c cannot be zero, because otherwise so would $\frac{a}{b}$ and we know that rational is non-zero. Thus $x = \frac{a}{b}\frac{d}{c} = \frac{ad}{bc}$. As ad and bc are integers with bc non-zero, this shows x is rational, a contradiction. $\square$

Chapter 3

Induction

3.1 Induction with Sums and Products

Use induction to prove the following statements.

1. For each $n \in \mathbb{N}$, $1+3+5+\cdots+(2n-3)+(2n-1)=n^2$.
2. For each $n \in \mathbb{N}$, $1+3+5+\cdots+(2n-1)+(2n+1)=(n+1)^2$.

   Proof: For the basis case of $n=1$ the left hand side is $1+3$ and the right hand side is $(1+1)^2$. These are equal.

   Now assume that
   $$1+3+5+\cdots+(2k+1)=(k+1)^2.$$
   We must show
   $$1+3+5+\cdots+(2k+1)+(2(k+1)+1)=(k+1+1)^2.$$

   $$1+3+5+\cdots+(2k+1)+(2(k+1)+1)=(k+1)^2+(2(k+1)+1)=$$

   $$(k+1)^2+(2(k+1)+1)=k^2+2k+1+2k+3=k^2+4k+4=(k+2)^2$$
   which is exactly what we needed to show. $\square$
3. For each $n \in \mathbb{N}$, $1+4+7+\cdots+(3n-2)=\frac{3n^2-n}{2}$.
4. For each $n \in \mathbb{N}$, $1+4+7+\cdots+(3n+1)+(3n+4)=\frac{1}{2}(n+2)(3n+5)$.

   Proof: For our basis case of $n=1$ we get $1+4+7=\frac{1}{2}\cdot 3\cdot 8$, which is true as both sides equal twelve.

   Next we assume that
   $$1+4+7+\cdots+(3k+1)+(3k+4)=\frac{1}{2}(k+2)(3k+5)$$

and will show

$$1+4+7+\cdots+(3k+1)+(3k+4)+(3(k+1)+4)=\frac{1}{2}(k+1+2)(3(k+1)+5).$$

Keeping in mind that the right hand side equals $\frac{1}{2}(k+3)(3k+8)$ we will start with the left hand side and confirm that the two are equal.

$$1+4+7+\cdots+(3k+1)+(3k+4)+(3(k+1)+4)=\frac{1}{2}(k+2)(3k+5)+(3(k+1)+4)=$$

$$\frac{1}{2}(k+2)(3k+5)+(3k+7)=\frac{1}{2}(k+2)(3k+5)+\frac{1}{2}(6k+14)=\frac{1}{2}[(k+2)(3k+5)+(6k+14)]=$$

$$\frac{1}{2}[3k^2+11k+10+6k+14]=\frac{1}{2}[3k^2+17k+24]=\frac{1}{2}(k+3)(3k+8).$$

$\square$

5. For each $n \in \mathbb{N}$, $2+5+8+\cdots+(3n-1)=\frac{3n^2+n}{2}$.

   Proof: In the basis case of $n=1$ the statement is $2=\frac{3\cdot 1+1}{2}$ which is true.

   Now we assume that

   $$2+5+8+\cdots+(3k-1)=\frac{3k^2+k}{2}$$

   and wish to show that

   $$2+5+8+\cdots+(3k-1)+(3(k+1)-1)=\frac{3(k+1)^2+(k+1)}{2}.$$

   The right hand side is simply $\frac{1}{2}(k+1)(3k+4)$ so we simplify the left and work towards this.

   $$2+5+8+\cdots+(3k-1)+(3(k+1)-1)=\frac{3k^2+k}{2}+(3(k+1)-1)=\frac{3k^2+k}{2}+(3k+2)=$$

   $$\frac{3k^2+k}{2}+\frac{6k+4}{2}=\frac{3k^2+7k+4}{2}=\frac{(k+1)(3k+4)}{2}. \quad \square$$

6. For each $n \in \mathbb{N}$, $1+5+9+\cdots+(4n-3)=2n^2-n$.

7. For each $n \in \mathbb{N}$, $2+6+10+\cdots+(4n-2)=2n^2$.

8. For each $n \in \mathbb{N}$, $1+6+11+\cdots+(5n-4)=\frac{5n^2-3n}{2}$.

   Proof: For our basis case of $n=1$ we get $1=\frac{5-3}{2}$ which is true.

   Next we assume $1+6+11+\cdots+(5k-4)=\frac{5k^2-3k}{2}$ and will use this to show that $1+6+11+\cdots+(5k-4)+(5(k+1)-4)=\frac{5(k+1)^2-3(k+1)}{2}$. Noting that the right hand side equals $\frac{5(k^2+2k+1)-3k-3}{2}=\frac{5k^2+10k+5-3k-3)}{2}=\frac{5k^2+7k+2}{2}$, we proceed to simply the left until we arrive at this formula.

$$1+6+11+\cdots+(5k-4)+(5(k+1)-4)=\frac{5k^2-3k}{2}+(5(k+1)-4)=$$
$$\frac{5k^2-3k}{2}+\frac{2}{2}(5k+1)=\frac{5k^2-3k+10k+2}{2}=\frac{5k^2+7k+2}{2}$$

$\square$

9. For each $n\in\mathbb{N}$, $6+12+18+\cdots+6n=3n^2+3n$.
10. For each $n\in\mathbb{N}$, $6+11+16+21+\cdots+(5n+1)=\frac{n(5n+7)}{2}$.
11. For each $n\in\mathbb{N}$, $1-2+3-4+\cdots+(2n-1)-2n+(2n+1)-(2n+2)=-(n+1)$.

    Proof: For a basis of $n=1$ the left hand side is $1-2+3-4$ and the right hand side is $-(1+1)$ which is equal.

    Now assume that

    $$1-2+3-4+\cdots+(2k-1)-2k+(2k+1)-(2k+2)=-(k+1)$$

    and we need to show that

    $$1-2+3-4+\cdots+(2k-1)-2k+(2k+1)-(2k+2)+(2(k+1)+1)-(2(k+1)+2)=-(k+1+1).$$

    Notice that

    $$1-2+3-4+\cdots+(2k-1)-2k+(2k+1)-(2k+2)+(2(k+1)+1)-(2(k+1)+2)=$$

    $$-(k+1)+(2(k+1)+1)-(2(k+1)+2)=-k-1+2k+2+1-2k-2-2=-k-2=-(k+1+1).$$

    $\square$

12. For each $n\in\mathbb{N}$, $1-3+5-7+\cdots+(2n-1)-(2n+1)=-2n$.
13. For each $n\in\mathbb{N}$, $(2n)^2-(2n-1)^2+(2n-2)^2-(2n-3)^2+\cdots+4^2-3^2+2^2-1^2=n(2n+1)$.

    Proof: For our basis case of $n=1$ the equation becomes $2^2-1^2=1(2+1)$ which is true.

    Now assume that we know

    $$(2k)^2-(2k-1)^2+(2k-2)^2-(2k-3)^2+\cdots+4^2-3^2+2^2-1^2=k(2k+1)$$

    and we will show that

    $$(2(k+1))^2-(2(k+1)-1)^2+(2k)^2-(2k-1)^2+(2k-2)^2-(2k-3)^2+\cdots+4^2-3^2+2^2-1^2$$
    $$=(k+1)(2(k+1)+1).$$

    $$(2(k+1))^2-(2(k+1)-1)^2+(2k)^2-(2k-1)^2+(2k-2)^2-(2k-3)^2+\cdots+4^2-3^2+2^2-1^2=$$

$$(2(k+1))^2-(2(k+1)-1)^2+k(2k+1)=(2k+2)^2-(2k+1)^2+k(2k+1)=$$

$$(4k^2+8k+4)-(4k^2+4k+1)+2k^2+k=2k^2+5k+3=(k+1)(2k+3)$$

which is equal to what we needed to arrive at. $\square$

14. For each $n \in \mathbb{N}$, $1^2+2^2+3^2+\cdots+(n-1)^2+n^2=\frac{n(n+(1/2))(n+1)}{3}$.

    Proof: Our basis case of $n=1$ gives us the equation $1^2=\frac{1\cdot(3/2)\cdot 2}{3}$ which is true.

    Now assume that
    $$1^2+2^2+3^2+\cdots+(k-1)^2+k^2=\frac{k(k+(1/2))(k+1)}{3}$$
    and we will show
    $$1^2+2^2+3^2+\cdots+(k-1)^2+k_+^2(k+1)^2=\frac{(k+1)(k+1+(1/2))(k+2)}{3}.$$
    The right hand side equals $\frac{1}{6}(k+1)(k+2)(2k+3)$ so we now proceed to simply the left until we arrive at this result.
    $$1^2+2^2+3^2+\cdots+(k-1)^2+k^2+(k+1)^2=\frac{k(k+(1/2))(k+1)}{3}+(k+1)^2=$$
    $$\frac{k(k+(1/2))(k+1)}{3}+\frac{3(k+1)^2}{3}=\frac{k(k+(1/2))(k+1)+3(k+1)^2}{3}=$$
    $$\frac{1}{3}(k(k+(1/2))(k+1)+3(k+1)^2)=\frac{1}{3}(k+1)(k(k+(1/2))+3k+3)=$$
    $$\frac{1}{3}(k+1)(k^2+\frac{7}{2}k+3)=\frac{1}{3}(k+1)\frac{1}{2}(2k^2+7k+6)=\frac{1}{6}(k+1)(2k+3)(k+2).$$
    $\square$

15. For each $n \in \mathbb{N}$, $1^3+2^3+3^3+\cdots+(n-1)^3+n^3=\left(\frac{n(n+1)}{2}\right)^2$.

16. For each $n \in \mathbb{N}$, $2+2^2+2^3+\cdots+2^{n-1}+2^n=2^{n+1}-2$.

17. For each $n \in \mathbb{N}$, $3+3^2+3^3+\cdots+3^{n-1}+3^n=\frac{1}{2}(3^{n+1}-3)$.

18. For each $n \in \mathbb{N}$ and $r \in \mathbb{R}-\{1\}$, $1+r^1+r^2+r^3+\cdots+r^{n-1}+r^n=\frac{r^{n+1}-1}{r-1}$.

    Proof: Our basis case of $n=1$ makes the equation $1+r=\frac{r^2-1}{r-1}$ which is true because $\frac{r^2-1}{r-1}=\frac{(r+1)(r-1)}{r-1}=r+1$.

    Assume now that
    $$1+r^1+r^2+r^3+\cdots+r^{k-1}+r^k=\frac{r^{k+1}-1}{r-1}$$
    and we will show
    $$1+r^1+r^2+r^3+\cdots+r^{k-1}+r^k+r^{k+1}=\frac{r^{k+1+1}-1}{r-1}.$$

$$1 + r^1 + r^2 + r^3 + \cdots + r^{k-1} + r^k + r^{k+1} = \frac{r^{k+1}-1}{r-1} + r^{k+1} = \frac{r^{k+1}-1}{r-1} + \frac{r^{k+1}(r-1)}{r-1} =$$

$$\frac{r^{k+1}-1+r^{k+1}(r-1)}{r-1} = \frac{r^{k+1}-1+r^{k+2}-r^{k+1}}{r-1} = \frac{r^{k+2}-1}{r-1}.$$

□

19. For each $n \in \mathbb{N}$, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$.

20. For each $n \in \mathbb{N}$, $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

21. For each $n \in \mathbb{N}$, $\frac{2}{1\cdot 3} + \frac{2}{2\cdot 4} + \frac{2}{3\cdot 5} + \cdots + \frac{2}{(n-1)(n+1)} + \frac{2}{n(n+2)} = \frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}$.

22. For each $n \in \mathbb{N}$, $n \in \mathbb{N}$, $\frac{3}{(1\cdot 2)^2} + \frac{5}{(2\cdot 3)^2} + \frac{7}{(3\cdot 4)^2} + \cdots + \frac{2n-1}{((n-1)n)^2} + \frac{2n+1}{(n(n+1))^2} = 1 - \frac{1}{(n+1)^2}$.

    Proof: When $n = 1$ notice the our left hand side equals $\frac{3}{(1\cdot 2)^2} = \frac{3}{4}$ which is also equal to $1 - \frac{1}{2^2}$. That gives us our basis case.

    Next we assume $\frac{3}{(1\cdot 2)^2} + \frac{5}{(2\cdot 3)^2} + \frac{7}{(3\cdot 4)^2} + \cdots + \frac{2k-1}{((k-1)k)^2} + \frac{2k+1}{(k(k+1))^2} = 1 - \frac{1}{(k+1)^2}$ and will show $\frac{3}{(1\cdot 2)^2} + \frac{5}{(2\cdot 3)^2} + \frac{7}{(3\cdot 4)^2} + \cdots + \frac{2k-1}{((k-1)k)^2} + \frac{2k+1}{(k(k+1))^2} + \frac{2k+3}{((k+1)(k+2))^2} = 1 - \frac{1}{(k+2)^2}$.

    Now by our assumption

$$\frac{3}{(1\cdot 2)^2} + \frac{5}{(2\cdot 3)^2} + \frac{7}{(3\cdot 4)^2} + \cdots + \frac{2k-1}{((k-1)k)^2} + \frac{2k+1}{(k(k+1))^2} + \frac{2k+3}{((k+1)(k+2))^2} =$$

$$1 - \frac{1}{(k+1)^2} + \frac{2k+3}{((k+1)(k+2))^2} = 1 - \frac{(k+2)^2}{(k+1)^2(k+2)^2} + \frac{2k+3}{((k+1)(k+2))^2} =$$

$$1 - \left(\frac{(k+2)^2}{(k+1)^2(k+2)^2} - \frac{2k+3}{((k+1)(k+2))^2}\right) = 1 - \frac{(k+2)^2 - 2k - 3}{((k+1)(k+2))^2} =$$

$$1 - \frac{k^2+4k+4-2k-3}{((k+1)(k+2))^2} = 1 - \frac{k^2+2k+1}{((k+1)(k+2))^2} = 1 - \frac{(k+1)^2}{((k+1)(k+2))^2} = 1 - \frac{1}{(k+2)^2}.$$

□

23. Prove that $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$ for all $n \in \mathbb{N}$.

24. Prove that $1(1!) + 2(2!) + 3(3!) + \cdots + (n-1)(n-1)! + n(n!) = (n+1)! - 1$ for all $n \in \mathbb{N}$.

25. For each $n \in \mathbb{N}$, $(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) \cdots (1 - \frac{1}{n})(1 - \frac{1}{n+1}) = \frac{1}{n+1}$.

26. For each $n \in \mathbb{N}$, $1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3) \cdot (2n-1) = \frac{(2n)!}{n!2^n}$.

    Proof: In the basis case of $n = 1$ the statement becomes $1 = \frac{2!}{1!\cdot 2}$ which is true.

    Next assume that $1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-3) \cdot (2k-1) = \frac{(2k)!}{k!2^k}$ and we will show that $1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-3) \cdot (2k-1) \cdot (2k+1) = \frac{(2(k+1))!}{(k+1)!2^{k+1}}$. The right hand side becomes $\frac{(2k)!(2k+1)(2k+2)}{k!(k+1)2^k \cdot 2}$ which equals $\frac{(2k)!}{k!2^k} \cdot \frac{(2k+1)(2k+2)}{2(k+1)}$

or simply $\frac{(2k)!}{k!2^k} \cdot (2k+1)$ so we start with the left and work towards this. Plugging in our assumption into the left gives us

$$1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k-3) \cdot (2k-1) \cdot (2k+1) = \frac{(2k)!}{k!2^k}(2k+1)$$

so we are immediately done. $\square$

3.2 Induction with Parity and Divisibility

Use induction to prove the following statements about divisibility over the integers.

1. Prove that $n(n+1)$ is always even for any natural number n.

2. Prove that $(n+1)(n+2)$ is always even for any natural number n.

3. If m is an even number, then m^n is even for any n in $\mathbb{N}$.

4. If m is an even number, then m^n is divisible by four for any n in $\mathbb{N}$.

   Proof: For our basis step note that if m is even then $m^1 = m$ which is even.

   Next we assume that m^k is even and we will show m^{k+1} is even.

   Notice that $m^{k+1} = m^k \cdot m$. As m^k is even and m is even we can write these as $2r$ and $2s$ for $r, s \in \mathbb{N}$. Thus $m^k \cdot m = 2r \cdot 2s = 4rs$. Setting $t = rs \in \mathbb{Z}$ shows this product is divisible by four. □

5. If m is an odd number, then m^n is odd for any n in $\mathbb{N}$.

6. If m is an even number, then for any n in $\mathbb{N}$ with $n > 1$, m^n is divisible by four.

7. Three divides $n^3 - n$ for all $n \in \mathbb{N}$.

   Proof: Note that when $n = 1$ our basis requires us to show three divides zero. This is true as three times zero is zero.

   Next assume that three divides $k^3 - k$ so for some $r \in \mathbb{Z}$, $3r = k^3 - k$. We must show that three divides $(k+1)^3 - (k+1)$. Now $(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = k^3 + 3k^2 + 2k = k^3 - k + (k + 3k^2 + 2k) = 3r + (3k^2 + 3k) = 3(r + k^2 + k)$. Set $s = r + k^2 + k \in \mathbb{Z}$. to get that $3s = (k+1)^3 - (k+1)$ thus showing $(k+1)^3 - (k+1)$ is divisible by three. □

8. Three divides $n^3 + 8n - 9$ for all $n \in \mathbb{N}$.

9. Six divides $n^3 - n$ for all $n \in \mathbb{N}$.

10. Five divides $n^5 - n$ for all $n \in \mathbb{N}$.

11. Four divides $(n^3 - n)(n+2)$ for all $n \in \mathbb{N}$.

    Proof: For our basis case consider that when $n = 1$, $(n^3 - n)(n+2)$ is zero, which is divisible by four.

    Next we assume that four divides $(k^3 - k)(k+2)$, and will show it divides $((k+1)^3 - (k+1))(k+1+2)$. We know $4r = (k^3 - k)(k+2) = k^4 + 2k^3 - k^2 - 2k$ for some $r \in \mathbb{Z}$. As

    $$((k+1)^3 - (k+1))(k+1+2) = k^4 + 6k^3 + 11k + 6k = (k^4 + 2k^3 - k^2 - 2k) + 4k^3 + 12k^2 + 8k =$$

    $$4r + 4k^3 + 12k^2 + 8k = 4(r + k^3 + 3k^2 + 2k)$$

    we can set $s = r + k^3 + 3k^2 + 2k$ to show $4s = ((k+1)^3 - (k+1))(k+1+2)$. As s is an integer, we know this expression is divisible by four, thus completing the proof. □

12. Six divides $7^n - 1$ for all $n \in \mathbb{N}$.

13. Four divides $5^n - 1$ for all $n \in \mathbb{N}$.

    Proof: Basis Step - 4 Divides $5^1 - 1$ since 4 divides 4.

    Inductive Step - Assume that 4 divides $5^k - 1$ and show that 4 divides $5^{k+1} - 1$. Since 4 divides $5^k - 1$ we can write $5^k - 1 = 4m$ for some $m \in \mathbb{Z}$.

    Now $5^{k+1} - 1 = 5 \cdot 5^k - 1 = 5 \cdot (5^k - 1 + 1) - 1 = 5 \cdot (4m + 1) - 1 = 5 \cdot 4m + 5 - 1 = 5 \cdot 4m + 4 = 4(5m + 1) = 4n$ for $n = 5m + 1 \in \mathbb{Z}$. This shows that $5^{k+1} - 1$ is a multiple of 4. $\square$

14. Three divides $2^{2n} - 1$ for all $n \in \mathbb{N}$.

    Proof: Basis step - 3 divides $2^2 - 1$ since three divides three.

    Inductive Step - Assume that 3 divides $2^{2k} - 1$ and therefore $3r = 2^{2k} - 1$ for some integer r. We will show that 3 divides $2^{2(k+1)} - 1$.

    Note that $2^{2(k+1)} - 1 = 2^{2k} \cdot 4 - 1$. Since $3r + 1 = 2^{2k}$ we know this equals $4(3r + 1) - 1$ which is $12r + 3$ or $3(4r + 1)$. Setting $s = 4r + 1 \in \mathbb{Z}$ completes the proof by showing $2^{2(k+1)} - 1 = 3s$. $\square$

15. For all $n \in \mathbb{N}$, $5^{2n} - 1$ is divisible by four.

16. For all $n \in \mathbb{N}$, $5^{2n} - 1$ is divisible by twenty-four.

17. For each $n \in \mathbb{N}$, $7^{4n} - 1$ is divisible by one-hundred.

    Proof: For our basis of $n = 1$ notice that $7^4 - 1 = 2400$ which is divisible by one-hundred.

    Next assume that $7^{4k} - 1$ is divisible by one-hundred and we will show that $7^{4(k+1)} - 1$ is as well. We know $100r = 7^{4k} - 1$ for some integer r. Now

$$7^{4(k+1)} - 1 = 7^{4k+4} - 1 = 7^{4k} \cdot 7^4 - 1 = (100r + 1) \cdot 7^4 - 1 =$$

$$100r \cdot 7^4 + 7^4 - 1 = 100r \cdot 7^4 + 2400 = 100(7^4 r + 24).$$

    We can set s to be the integer $7^4 r + 24$ to show that $100s = 7^{4(k+1)} - 1$, which completes the proof by showing one-hundred divides $7^{4(k+1)} - 1$. $\square$

18. For each $n \in \mathbb{N}$, $4^{3n} - 1$ is divisible by nine.

19. Three divides $5^{2n} + 2$ for all $n \in \mathbb{N}$.

    Basis Step - 3 divides $5^2 + 2 = 27$ since $27 = 3 \cdot 9$

    Inductive Step: Assume that 3 divides $5^{2k} + 2$ and show that 3 divides $5^{2(k+1)} + 2$.

    Since 3 divides $5^{2k} + 2$ we can write $5^{2k} + 2 = 3m$ for some $m \in \mathbb{Z}$. Now

$$5^{2(k+1)} + 2 = 5^{2k+2} + 2 = 5^{2k} \cdot 5^2 + 2 = (5^{2k} + 2 - 2) \cdot 5^2 + 2 =$$

$$(3m - 2) \cdot 5^2 + 2 = 3m \cdot 5^2 - 2 \cdot 5^2 + 2 = 3m \cdot 5^2 - 48 = 3(m \cdot 5^2 - 16) = 3l$$

    for $l = (m \cdot 5^2 - 16) \in \mathbb{Z}$. $\square$

20. For all n in $\mathbb{N}$, seven divides $3^{6n} + 6$.

21. For all n in $\mathbb{N}$, three divides $5^n - 2^n$.

22. For all n in $\mathbb{N}$, four divides $7^n - 3^n$.

23. For all n in $\mathbb{N}$, seven divides $11^n - 4^n$.

    Proof: For our basis of $n = 1$ notice that seven divides $11^1 - 4^1 = 7$ because seven times one equals seven.

    Next we assume that seven divides $11^k - 4^k$ and thus $7r = 11^k - 4^k$ for some integer r. We wish to show that seven divides $11^{k+1} - 4^{k+1}$. Now $11^{k+1} - 4^{k+1} = 11 \cdot 11^k - 4 \cdot 4^k$. We know $11^k = 7r + 4^k$ so our expression equals

    $$11(7r + 4^k) - 4 \cdot 4^k = 7 \cdot 11r + 11 \cdot 4^k - 4 \cdot 4^k = 7 \cdot 11r + 7 \cdot 4^k = 7(11r + 7 \cdot 4^k).$$

    Setting $s = 11r + 7 \cdot 4^k \in \mathbb{Z}$ we get $11^{k+1} - 4^{k+1} = 7s$, which shows it to be divisible by seven. $\square$

24. For any number bigger than three, we can provide exact postage in 2 and 5 cent stamps. [This is the same as asking to show $2r + 5s = n$ has a solution in integers for every n greater than three[1].

25. For any number bigger than seven, we can provide exact postage in 3 and 5 cent stamps. [This is the same as asking to show $3r + 5s = n$ has a solution in integers for every $n > 7$.

    Proof: For our basis step note that when $n = 8$ we have $3(1) + 5(1) = 8$.

    Now assume that we can solve $3r + 5s = k$ and we must show that we can solve $3t + 5u = k + 1$ with $r, s, t, u \in \mathbb{Z}$.

    If $s > 0$ then we can take $3(r + 2) + 5(s - 1)$ which equals $3r + 6 + 5s - 1 = (3r + 5s) + 1 = k + 1$.

    If $s = 0$ then we know r is at least 3 because $3r = k$ and $k > 6$. Thus we can subtract three from r and add two to s. This gives is $3(r - 3) + 5(s + 2) = (3r + 5s) - 9 + 10 = k + 1$. $\square$

26. For any natural number n greater than two, $2n$ divides $n!$.

    Proof: For our basis, note that when n is three both $2n$ and $n!$ equal six. As six divides itself, this completes this part of the proof.

    Next we assume $2k$ divides $k!$ and thus $2km = k!$ for some $m \in \mathbb{Z}$. We wish to show $2(k + 1)$ divides $(k+1)!$. We know $(k+1)! = k!(k+1) = 2km(k+1) = 2(k+1)(mk)$. Set r to be the integer mk to see that $(k + 1)! = 2(k + 1)r$, thus showing $2(k + 1)$ divides $(k + 1)!$. $\square$

27. For any natural number n greater than three, $6n$ divides $n!$.

28. For all $n \in \mathbb{N}$, prove that 2^n divides $(2n)!$.

    Proof: For our basis of $n = 1$ we simply note that two divides two.

    Next assume that 2^k divides $(2k)!$ and thus for some integer r, $2^k r = (2k)!$. We must show 2^{k+1} divides $(2(k + 1))!$. Note that

    $$(2(k + 1))! = (2k + 2)! = (2k + 1)!(2k + 2) = (2k)!(2k + 1)(2k + 2) =$$

    $$2^k r(2k + 1)(2k + 2) = 2^k r(2k + 1)2(k + 1) = 2^{k+1} r(2k + 1)(k + 1).$$

    We set s to be the integer $r(2k+1)(k+1)$ to show that $(2(k+1))! = 2^{k+1}s$ and thus prove $(2(k+1))!$ is divisible by 2^{k+1}. $\square$

[1] Though this technically isn't a straight out divisibility question, we are trying to show that n is the sum of a number divisible by 2 and a number divisble by 5. This is why this question is placed in this section.

29. The product of n odd numbers is also an odd number for any n in $\mathbb{N}$.

30. The product of n even numbers is also an even number for any n in $\mathbb{N}$.

31. The sum of $2n$ odd numbers is an even number for any n in $\mathbb{N}$.

32. The sum of $2n-1$ odd numbers is an odd number for any n in $\mathbb{N}$.

    Proof: For our basis, notice that when there is just $2(1)-1=1$ odd, the total is odd.

    Assume the sum of $2k-1$ odd numbers is an odd number and we will show that the sum of $2k+1$ odd numbers is an odd number.

    If we add $2k+1$ odds we get a sum of the form $a_1+a_2+\cdots+a_{2k-1}+a_{2k}+a_{2k+1}$. The first $2k-1$ of these sum to an odd by our inductive step, so we have $b+a_{2k}+a_{2k+1}$.

    We can write the sum as $(2r+1)+(2s+1)+(2t+1)$ for $r,s,t\in\mathbb{Z}$ which gives us $2(r+s+t)+3=2(r+s+t+1)+1$. Setting $u=r+s+t+1\in\mathbb{Z}$ shows us this is odd. $\square$

3.3 Induction with Inequalities

Use induction to prove the following statements. Recall that these problems tend to involve giving up a certain amount of information at one step, since you rarely immediately get the outcome you want. Because of this there are a variety of ways to solve these. Each problem also has two directions you can work in.

1. For all $n \in \mathbb{N}$ with $n > 3$, $\frac{5}{3}n + 1 > \frac{4}{5}n + 4$[2].

   Proof: In a basis case of $n = 4$ we get $\frac{20}{3} + 1 = \frac{23}{3} = \frac{115}{15} > \frac{108}{15} = \frac{36}{5} = \frac{16}{5} + 4$.

   Next we assume that $\frac{5}{3}k + 1 > \frac{4}{5}k + 4$ and we will show that $\frac{5}{3}(k+1) + 1 > \frac{4}{5}(k+1) + 4$. We know

$$\frac{5}{3}(k+1) + 1 = \frac{5}{3}k + \frac{5}{3} + 1 = \left(\frac{5}{3}k + 1\right) + \frac{5}{3} > \frac{4}{5}k + 4 + \frac{5}{3} >$$

$$\frac{4}{5}k + 4 + \frac{4}{5} = \frac{4}{5}k + \frac{4}{5} + 4 = \frac{4}{5}(k+1) + 4$$

   which completes our proof. □

2. For all $n \in \mathbb{N}, n^2 \geq 2n - 1$

3. For all $n \in \mathbb{N}$ with $n \geq 3, n^2 \geq 3n$

4. For all $n \in \mathbb{N}$ with $n \geq 3, n^2 > 2n + 1$

5. For all $n \in \mathbb{N}$ with $n \geq 2, n^3 > 3n + 1$

   Proof: The basis case of $n = 2$ is given by the fact that $2^3 = 8 > 7 = 3(2) + 1$.

   Assume now that $k^3 > 3k + 1$. We must show $(k+1)^3 > 3(k+1) + 1 = 3k + 4$. Then

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1 > (3k+1) + (3k^2 + 3k + 1) = 3k^2 + 6k + 2 > 3k + 2 + (3k^2 + 3k) >$$

$$3k + 2 + (3(2)^2 + 3(2)) = 3k + 2 + 18 > 3k + 4.$$

   □

6. For all $n \in \mathbb{N}, n^3 \geq 3(n^2 - n)$

   Proof: The basis case of $n = 1$ is taken care of by the fact that $1^3 = 1 \geq 0 = 3(1^2 - 1)$.

   Assume that $k^3 \geq 3(k^2 - k)$. We wish to show $(k+1)^3 \geq 3((k+1)^2 - (k+1)) = 3k^2 + 3k$. Notice that

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1 \geq 3(k^2 - k) + 3k^2 + 3k + 1 =$$

$$6k^2 + 1 = 3k^2 + (3k^2) + 1 \geq 3k^2 + 3k$$

   as $k^2 > k$ on $\mathbb{N}$. □

[2]Note that this proof and a few others here are doable easily without induction, however they are still good for practicing induction.

7. For all $n \in \mathbb{N}$ with $n \geq 2, n^4 \geq 2n^3$.

   Proof: For the basis case of $n = 2$ notice $2^4 = 16 \geq 2(2)^3$.

   Next assume that $k^4 \geq 2k^3$. To complete the proof we must show $(k+1)^4$ is greater or equal than $2(k+1)^3$ which equals $2k^3 + 6k^2 + 6k + 2$. Then

   $$(k+1)^4 = k^4 + 4k^3 + 6k^2 + 4k + 1 \geq 2k^3 + 4k^3 + 6k^2 + 4k + 1 =$$

   $$6k^3 + 6k^2 + 4k + 1 = 2k^3 + 6k^2 + 4k + 1 + (4k^3).$$

   Since $4k^3 = 2k^3 + 2k^3 \geq 2k + 1$ we know our expression is bigger than $2k^3 + 6k^2 + 4k + 1 + (2k+1) = 2k^3 + 6k^2 + 6k + 2$. □

8. For all $n \in \mathbb{N}, 2^n + 1 \leq 3^n$.

9. For all $n \in \mathbb{N}$ with $n \geq 2, 2^n + 5 \leq 3^n$.

   Proof: Our basis case here is $n = 2$, which gives us $9 \leq 9$.

   Assume now that $2^k + 5 \leq 3^k$ and we will show $2^{k+1} + 5 \leq 3^{k+1}$. Now

   $$2^{k+1} + 5 = 2 \cdot 2^k + 5 \leq 2(3^k - 5) + 5 = 2 \cdot 3^k - 10 + 5 =$$

   $$2 \cdot 3^k - 5 < 2 \cdot 3^k < 3 \cdot 3^k = 3^{k+1}$$

   . □

10. For all $n \in \mathbb{N}$ with $n \geq 2, 3^n + 7 \leq 4^n$.

11. For all $n \in \mathbb{N}$, $\left(\frac{3}{2}\right)^n \geq n + \left(\frac{1}{2}\right)^n$.

12. Given $a, n \in \mathbb{N}$, $\left(\frac{a+2}{2}\right)^n \geq n + \left(\frac{a}{2}\right)^n$.

    Proof: Let a be any fixed natural and we will use a proof by induction on n. When $n = 1$ we know $\frac{a+2}{2} = \frac{a}{2} + \frac{2}{2} = 1 + \frac{a}{2}$ so our statement is true.

    Next we assume that $\left(\frac{a+2}{2}\right)^k \geq k + \left(\frac{a}{2}\right)^k$ and will show $\left(\frac{a+2}{2}\right)^{k+1} \geq (k+1) + \left(\frac{a}{2}\right)^{k+1}$.

    Now

    $$\left(\frac{a+2}{2}\right)^{k+1} = \frac{a+2}{2}\left(\frac{a+2}{2}\right)^k = \frac{a}{2}\left(\frac{a+2}{2}\right)^k + \frac{2}{2}\left(\frac{a+2}{2}\right)^k \geq \frac{a}{2}(k + \left(\frac{a}{2}\right)^k\Big) + k + \left(\frac{a}{2}\right)^k =$$

    $$\frac{a}{2}k + \left(\frac{a}{2}\right)^{k+1} + k + \left(\frac{a}{2}\right)^k = \left(\frac{a}{2}\right)^{k+1} + k + \left(\frac{a}{2}k + \left(\frac{a}{2}\right)^k\right)$$

    Now the term $(\frac{ak}{2} + (\frac{a}{2})^k)$ is always at least one. When $k = a = 1$ it equals one, and otherwise $\frac{ak}{2}$ alone is at least $\frac{2}{2}$ which equals one. Thus $(\frac{a}{2})^{k+1} + k + (\frac{a}{2}k + (\frac{a}{2})^k) \geq (\frac{a}{2})^{k+1} + k + 1$, completing our proof. □

13. Given $a, n \in \mathbb{N}$, $a^n + 2a + 1 \leq (a+1)^n$ for all $n \geq 2$.

    Proof: For our basis case we must show $a^2 + 2a + 1 \leq (a+1)^2$, which is true because the sides are equal.

    Assume that $a^k + 2a + 1 \leq (a+1)^k$ and we must show $a^{k+1} + 2a + 1 \leq (a+1)^{k+1}$. We know

$$a^{k+1} + 2a + 1 = a \cdot a^k + 2a + 1 = a \cdot a^k + 2a + 1 \leq a \cdot ((a+1)^k - 2a - 1) + 2a + 1 =$$

$$a(a+1)^k - 2a^2 - 1 + 2a = a(a+1)^k - 2a(a-1) - 1.$$

    Noting both a and $a-1$ are positive shows us $-2a(a-1) - 1$ is negative and thus

$$a(a+1)^k - 2a(a-1) - 1 \leq a(a+1)^k < (a+1)(a+1)^k = (a+1)^{k+1}.$$

□

14. For all $n \in \mathbb{N}$ with $n \geq 4, 2^n \geq n^2$.

    Proof: For our basis case of $n = 4$ note that $2^4 = 16 = 4^2$.

    Next assume that $2^k \geq k^2$ and we will show $2^{k+1} \geq (k+1)^2$. We have

$$2^{k+1} = 2 \cdot 2^k \geq 2k^2 = k^2 + k^2 > k^2 + 3k = k^2 + 2k + k > k^2 + 2k + 1 = (k+1)^2.$$

    Here we used that $k > 3$ and $k > 1$ respectively in order to get our inequalities. □

15. For all $n \in \mathbb{N}$ with $n \geq 10, 2^n \geq n^3$ [Hint: Feel free to use that $n^2 > 2n + 1$ for $n \geq 3$ from a previous problem on this sheet.]

16. For all $n \in \mathbb{N}$ with $n \geq 3, n! > (\frac{3}{2})^n$

17. For all $n \in \mathbb{N}$ with $n \geq 4, n! > 2^n$.

18. For all $n \in \mathbb{N}$ with $n \geq 4, n! > n^2$.

    Proof: For our basis case, note that when $n = 4$, $n! = 24 > 9 = n^2$.

    Next assume that $k! > k^2$ and we will show $(k+1)! > (k+1)^2$. We know

$$(k+1)! = (k!)(k+1) = k \cdot k! + k! > k \cdot k^2 + k^2 = k^2 + k^3.$$

    As $k^3 > 4 \cdot 1 \cdot k = 2k + 2k > 2k + 1$ we get that $k^2 + k^3 > k^2 + 2k + 1 = (k+1)^2$ which completes our argument. □

    Alternate proof: We can use the basis case as before, and still assume $k! > k^2$, but start with $(k+1)^2$. We see that

$$(k+1)^2 = k^2 + 2k + 1 \leq k! + 2k + 1 < k! + 2k + k = k! + 3k < k! + k!k = k!(k+1) = (k+1)!.$$

    We used the fact that $1 < k$ and that $3 < k!$, both of which are true as we are starting at four. □

19. For all $n \in \mathbb{N}$ with $n \geq 12$ we have $n! > 5^n$.

20. For all $n \in \mathbb{N}$ with $n \geq 5$ we have $n! > 3^{n-1}$.

21. For all $n \in \mathbb{N}$ with $n \geq 5$ we have $(n!)^2 > 5^n$.

    Proof: For our basis case of $n = 5$ we get $(120)^2 = 14400 > 3125 = 5^5$.

    Now assume that $(k!)^2 > 5^k$ and we want to show $(k+1!)^2 > 5^{k+1}$. We know

    $$(k+1!)^2 = (k!(k+1))^2 = k!k!(k+1)^2 > 5^k \cdot 5^k \cdot (k+1)^2 > 5^k \cdot 5^k > 5^k \cdot 5 = 5^{k+1}.$$

    □

22. For all $n \in \mathbb{N}$ with $n \geq 4$ we have $(n!)^3 > 5^n$.

23. For all $n \in \mathbb{N}$ with $n \geq 4$ we have $30(n!) > 5^n$.

24. For all $n \in \mathbb{N}, n! \leq n^n$.

    Proof: For our basis of $n = 1$ note that $1! = 1 = 1^1$.

    Now assume $k! \leq k^k$ and we will show $(k+1)! \leq (k+1)^{(k+1)}$. We have

    $$(k+1)! = k! \cdot k \leq k^k \cdot k < (k+1)^k(k+1) = (k+1)^{(k+1)}$$

    . □

25. For all $n \in \mathbb{N}, 2^n + n! \leq (n+2)!$

26. For all $n \in \mathbb{N}, 3^n + n! \leq (n+3)!$

    Proof: When $n = 1$ we get $3 + 1 \leq 4!$ which is true.

    Next we assume $3^k + k! \leq (k+3)!$ and wish to show $3^{k+1} + (k+1)! \leq (k+4)!$. Knowing $3^k \leq (k+3)! - k!$ we can see

    $$3^{k+1} + (k+1)! = 3 \cdot 3^k + (k+1)! \leq 3(k+3)! - 3k! + (k+1)! \leq$$

    $$3(k+3)! + (k+1)! \leq 3(k+3)! + (k+3)! = 4(k+3)! < (k+4)(k+3)! = (k+4)!$$

    . □

27. For all $n \in \mathbb{N}$ with $n \geq 6, 4^n + n! \leq (n+1)!$

    Proof: When $n = 6$ we have a basis case of $4^n + n! = 4816 \leq= 5040 = (n+4)!$

    Next assume that $4^k + k! \leq (k+1)!$ and we will show $4^{k+1} + (k+1)! \leq (k+2)!$. We have

    $$4^{k+1} + (k+1)! = 4 \cdot 4^k + (k+1)! \leq$$

    $$4((k+1)! - k!) + (k+1)! = 5(k+1)! - k! < 5(k+1)!$$

    As $k \leq 6$ we know $5 < k+2$ and thus $5(k+1)! < (k+2)(k+1)! = (k+2)!$ completing the proof. □

28. For all a and n in $\mathbb{N}, a^n + n! \leq (a+3)!$

29. If a is a positive real constant then for all $n \in \mathbb{N}, (1+a)^n \geq 1 + na$.

30. For all $n \in \mathbb{N}, 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq \frac{n+2}{2}$

Proof: With our basis case of $n = 1$ we get $1 < \frac{3}{2}$.

Next assume $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \leq \frac{k+2}{2}$ and we will show $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \frac{1}{k+1} \leq \frac{k+3}{2}$. Substituting our inductive hypothesis gives us that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \frac{1}{k+1} \leq \frac{k+2}{2} + \frac{1}{k+1} = \frac{k+2+\frac{2}{k+1}}{2}.$$

This last fraction is less than or equal to three because $\frac{2}{k+1}$ is less than or equal to one. $\square$

3.4 Induction with Recurrence Relations

For the following questions, recall that the Fibonacci Numbers are given by the recurrence relation

$$f_1 = f_2 = 1, f_n = f_{n-1} + f_{n-2}.$$

Define the Jacobsthal numbers by setting $J_1 = J_2 = 1$ and using the recurrence

$$J_{n+2} = J_{n+1} + 2J_n.$$

1. Let r be any positive real number and define $a_0 = 1, a_n = ra_{n-1}$. Prove that $a_n = r^n$ for all $n \in \mathbb{N}$.

   Proof: $a_0 = 1 = r^0$, which gives us our basis case.

   Assume $a_k = r^k$ and we must show $a_{k+1} = r^{k+1}$. $a_{k+1} = ra_k = r \cdot r^k = r^{k+1}$ so we are done. $\square$

2. Let c be any positive real number and define $a_1 = c, a_n = \frac{1}{2}(c + a_{n-1})$. Prove that $0 \leq a_n \leq c$ for all $n \in \mathbb{N}$.

   Proof: As $a_1 = c \leq c$ we have our basis case.

   Next assume $a_k \leq c$ and we will show $a_{k+1} \leq c$. Note that $a_{k+1} = \frac{1}{2}(c + a_k) \leq \frac{1}{2}(c + c) = \frac{2c}{2} = c$ which completes the proof. $\square$

3. Let $a_1 = 1$ and $a_n = na_{n-1}$. Prove that $a_n = n!$ for all $n \in \mathbb{N}$.

4. Let $a_1 = 1$ and $a_n = a_{n-1} + 2n - 1$. Prove that $a_n = n^2$ for all $n \in \mathbb{N}$.

5. Let $a_1 = 1$ and $a_n = a_{n-1} + \frac{1}{2^{n-1}}$. Prove that $a_n = 2(1 - \frac{1}{2^n})$ for all $n \in \mathbb{N}$.

6. Let r be any real number not equal to one. Let $a_1 = 1$ and $a_n = a_{n-1} + r^{n-1}$. Prove that $a_n = \frac{1-r^n}{1-r}$ for all $n \in \mathbb{N}$.

   Proof: For our basis note that when $n = 1$ we have $a_1 = 1$ which equals $\frac{1-r^1}{1-r}$ though only because $r \neq 1$.

   Next assume $a_k = \frac{1-r^k}{1-r}$ and we will show $a_{k+1} = \frac{1-r^{k+1}}{1-r}$. We know

   $$a_{k+1} = a_k + r^k = \frac{1 - r^k}{1 - r} + r^k = \frac{1 - r^k}{1 - r} + \frac{r^k(1 - r)}{1 - r} = \frac{1 - r^k}{1 - r} + \frac{r^k - r^{k+1}}{1 - r} = \frac{1 - r^{k+1}}{1 - r}.$$

   $\square$

7. Consider the recurrence given by $a_{n+2} = 2a_{n+1} + 3a_n$ with $a_1 = a_2 = 1$. Prove that for any $n \in \mathbb{N}$, $a_n \leq \frac{3}{2}(3^n)$.

   Proof: For our basis cases note that when $n = 1$ we have $1 \leq \frac{9}{2}$ and when $n = 2$ we get $1 \leq \frac{27}{2}$.

   Next assume that $a_k \leq \frac{3}{2}(3^k)$ and we will show $a_{k+1} \leq \frac{3}{2}(3^{k+1})$. Now $a_{k+1} = 2a_k + 3a_{k-1} \leq 2 \cdot \frac{3}{2}(3^k) + 3 \cdot \frac{3}{2}(3^{k-1}) = \frac{2}{2}(3^{k+1}) + \frac{1}{2}(3^{k+1}) = \frac{3}{2}(3^{k+1})$. $\square$

8. Consider the recurrence given by $a_{n+2} = 2a_{n+1} + 3a_n$ with $a_1 = a_2 = 1$. Prove that for any $n \in \mathbb{N}$, $a_n = \frac{1}{2}(3^n + (-1)^n)$.

9. Consider the recurrence given by $a_{n+2} = 2a_{n+1} + 3a_n$ with $a_1 = 0$ and $a_2 = 1$. Prove that for any $n \in \mathbb{N}$, $a_n \leq \frac{3}{4}(3^n)$.

10. Consider the recurrence given by $a_{n+2} = 2a_{n+1} + 3a_n$ with $a_1 = 0$ and $a_2 = 1$. Prove that for any $n \in \mathbb{N}$, $a_n = \frac{1}{4}(3^n - (-1)^n)$.

11. Prove that f_n is positive for any n.

12. Prove that $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.

13. Prove that $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$.

14. Prove that $f_2 + f_4 + \cdots + f_{2n} = f_{2n+1} - 1$.

15. Prove that $f_1 + f_4 + \cdots + f_{3n-2} = \frac{1}{2} f_{3n}$.

16. Prove that $f_2 + f_5 + \cdots + f_{3n-1} = \frac{1}{2}(f_{3n+1} - 1)$.

    Proof: For our basis case note that the statement holds for n equal to 1 as $f_2 = 1 + \frac{1}{2}(3-1) = \frac{1}{2}(f_4 - 1)$.

    Next assume $f_2 + f_5 + \cdots + f_{3n-1} = \frac{1}{2}(f_{3n+1} - 1)$ and we will show that $f_2 + f_5 + \cdots + f_{3n-1} + f_{3n+2} = \frac{1}{2}(f_{3n+4} - 1)$. We know by our assumption that

$$f_2 + f_5 + \cdots + f_{3n-1} + f_{3n+2} = \frac{1}{2}(f_{3n+1} - 1) + f_{3n+2} =$$

$$\frac{1}{2}(f_{3n+1} - 1 + 2f_{3n+2}) = \frac{1}{2}(f_{3n+1} + f_{3n+2} + f_{3n+2} - 1) = \frac{1}{2}(f_{3n+3} + f_{3n+2} - 1) = \frac{1}{2}(f_{3n+4} - 1).$$

$\square$

17. Prove that $f_3 + f_6 + \cdots + f_{3n} = \frac{1}{2}(f_{3n+2} - 1)$.

18. Prove that $f_{n+2} + f_{n-2} = 3f_n$ for all $n \geq 3$.

    Proof: For our basis case, note that the statement holds for n equal to 3 as $f_5 + f_1 = 5 + 1 = 6 = 3f_3$.

    Now assume that $f_{m+2} + f_{m-2} = 3f_m$ for all $m \leq k$ and we will show $f_{k+3} + f_{k-1} = 3f_{k+1}$. We know

$$f_{k+3} + f_{k-1} = f_{k+2} + f_{k+1} + f_{k-2} + f_{k-3} = f_{k+2} + f_{k-2} + f_{k+1} + f_{k-3}.$$

    Using our assumption for $m = k$ and $m = k-1$ we get $3f_k + 3f_{k-1}$ which equals $3(f_k + f_{k+1}) = 3f_{k+2}$. $\square$

19. Prove that $f_{n+6} - f_n = 4f_{n+3}$.

20. Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$.

21. Prove that $f_1 f_2 + f_2 f_3 + \cdots + f_{2n-1} f_{2n} = f_{2n}^2$.

    Proof: When $n = 1$ notice that $f_1 f_2 = f_2^2$ since $1 \cdot 1 = 1^2$. When $n = 2$ we have $f_1 f_2 + f_2 f_3 + f_3 f_4 = f_4^2$ since $1 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 = 3^2$[3].

[3] Since the last term in the sum must start with f_{2n-1}, when $n = 2$ we must stop at $f_3 f_4$. This is why we get three terms with only $n = 2$.

Next assume that $f_1f_2 + f_2f_3 + \cdots + f_{2m-1}f_{2m} = f_{2m}^2$ for all $m \leq k$, and we will show that $f_1f_2 + f_2f_3 + \cdots + f_{2m-1}f_{2m} + f_{2m}f_{2m+1} + f_{2m+1}f_{2m+2} = f_{2(m+1)}^2$. We get

$$f_1f_2 + f_2f_3 + \cdots + f_{2m-1}f_{2m} + f_{2m}f_{2m+1} + f_{2m+1}f_{2m+2} =$$

$$f_{2m}^2 + f_{2m}f_{2m+1} + f_{2m+1}f_{2m+2} = f_{2m}(f_{2m} + f_{2m+1}) + f_{2m+1}f_{2m+2} =$$

$$f_{2m}f_{2m+2} + f_{2m+1}f_{2m+2} = (f_{2m} + f_{2m+1})f_{2m+2} = f_{2m+2}f_{2m+2} = f_{2m+2}^2.$$

□

22. Prove that $f_1f_2 + f_2f_3 + \cdots + f_{2n}f_{2n+1} = f_{2n+1}^2 - 1$.

23. Prove that $f_n^2 = f_{n+1}f_{n-1} - (-1)^n$ for all integers $n \geq 2$.

    Proof: For our basis cases, we can check $n = 2$ and $n = 3$ to see $f_2^2 = 1 = 2 \cdot 1 - 1 = f_3f_1 - (-1)^2$ and $f_3^2 = 4 = 3 \cdot 1 + 1 = f_4f_2 - (-1)^3$.

    Now assume that $f_m^2 = f_{m+1}f_{m-1} - (-1)^m$ for all integers $2 \leq m \leq k$. and we will show $f_k^2 = f_{k+1}f_{k-1} - (-1)^k$. Since

    $$f_{k+1}^2 = (f_k + f_{k-1})^2 = f_k^2 + 2f_kf_{k-1} + f_{k-1}^2$$

    , using our assumption on $m = k - 1$ gives us

    $$f_k^2 + 2f_kf_{k-1} + f_kf_{k-2} - (-1)^{k-1} = f_k(f_k + 2f_{k-1} + f_{k-2}) - (-1)^{k-1} =$$

    $$f_k((f_k + f_{k-1}) + (f_{k-1} + f_{k-2})) - (-1)^{k-1} = f_k(f_{k+1} + f_k) - (-1)^{k-1} = f_kf_{k+2} - (-1)^{k-1}.$$

    Since $(-1)^{k-1} = (-1)^{k+1}$ for any integer k, this is exactly what we needed to show. □

24. Prove that $f_{2n} = f_{n+1}^2 - f_{n-1}^2$.

    Proof: For our basis cases, we can check $n = 2$ and $n = 3$ to see $f_4 = 3 = 4 - 1 = f_3^2 - f_1^2$ and $f_6 = 8 = 3^2 - 1^2 = f_4^2 - f_2^2$.

    Next assume the statement is true for all $m \leq k$ and we will show $f_{2(k+1)} = f_{k+2}^2 - f_k^2$ which is equal to $f_{n+1}^2 + 2f_nf_{n+1}$. Now

    $$f_{2(k+1)} = f_{2k+2} = f_{2k+1} + f_{2k} = 2f_{2k} + f_{2k-1} = 3f_{2k} - f_{2k-2} = 3f_{2k} - f_{2(k-1)}.$$

    Now our indices are both even so we can apply our assumption twice to get

    $$3f_{2k} - f_{2(k-1)} = 3f_{k+1}^2 - 3f_{k-1}^2 - f_k^2 + f_{k-2}^2 =$$

    $$f_{k+1}^2 + 2f_nf_{n+1} + (2f_{n+1}^2 - 3f_{n-1}^2 - 2f_nf_{n+1} - f_n^2 + f_{n-2}^2).$$

    If we can show the term in parenthesis is zero, we are done, but this is possible by replacing each f_{n+1} by $f_n + f_{n-1}$ and the f_{n-2} with $f_n - f_{n-1}$. □

25. Prove that for any $n \in \mathbb{N}$, f_{3n} is even and f_{3n-1} and f_{3n-2} are odd.

    Proof: For our basis note that $f_3 = 2$ which is even but f_2 and f_1 are one which is odd.

    Now assume that f_{3k} is even and f_{3k-1} and f_{3k-2} are odd. We must show that $f_{3(k+1)} = f_{3k+3}$ is even, and that f_{3k+2} and f_{3k+1} are odd. First f_{3k+1} is $f_{3k} + f_{3k-1}$ which we know by our assumption is an even plus an odd, hence odd. Similarly f_{3k+2} is $f_{3k+1} + f_{3k}$ which we now know is an odd plus an even, hence also odd. Finally $f_{3k+3} = f_{3k+2} + f_{3k+1}$ both of which we have shown to be odd, hence this sum is even, completing the proof. □

26. Find the flaw in the following proof that claims to show that f_n is even for all $n \geq 3$.

    "Proof:" When $n = 3$ we know $f_n = 2$ which is even.

    Now assume that f_m is even for all $m \leq k$ and we will show f_{k+1} is even. $f_{k+1} = f_k + f_{k-1}$ which is a sum of two evens by our assumption. Therefore f_{k+1} is even. $\square$

27. For all natural numbers n, prove that $J_n \leq \frac{1}{2} \cdot 2^n$ for all n.

    Proof: For our basis cases we have $J_1 = 1 \leq \frac{1}{2}(2^1)$ and $J_2 = 1 \leq 2 = \frac{1}{2}(2^2)$.

    Now we assume that $J_m \leq \frac{1}{2}(2^m)$ for all $m \leq k$ and will show $J_{k+1} \leq \frac{1}{2}(2^{k+1})$. We have

$$J_{k+1} = J_k + 2J_{k-1} \leq \frac{1}{2}2^k + 2\frac{1}{2}2^{k-1} = \frac{1}{2}(2^k + 2 \cdot 2^{k-1})$$

$$= \frac{1}{2}(2 \cdot 2^k) = \frac{1}{2}2^{k+1}.$$

$\square$

28. For all natural numbers n, prove that $J_n \geq \frac{1}{4} \cdot 2^n$ for all n.

    Proof: For our basis cases we have $J_1 = 1 \geq \frac{1}{2} = \frac{1}{4}(2^1)$ and $J_2 = 1 = \frac{1}{4}(2^2)$.

    Now we assume that $J_m \geq \frac{1}{4}(2^m)$ for all $m \leq k$ and will show $J_{k+1} \geq \frac{1}{4}(2^{k+1})$. We have

$$J_{k+1} = J_k + 2J_{k-1} \geq \frac{1}{4}2^k + 2\frac{1}{4}2^{k-1} = \frac{1}{4}(2^k + 2 \cdot 2^{k-1})$$

$$= \frac{1}{4}(2 \cdot 2^k) = \frac{1}{4}2^{k+1}.$$

$\square$

29. For all $n \in \mathbb{N}$, prove that $J_n = \frac{1}{3}(2^n - (-1)^n)$.

    Proof: When $n = 1$ we have $J_1 = 1 = \frac{2+1}{3} = \frac{1}{3}(2^1 - (-1)^1)$ and when $n = 2$ we have $J_2 = 1 = \frac{4-1}{3} = \frac{1}{3}(2^2 - (-1)^2)$, both of which are true. This gives us our basis cases.

    Next assume that $J_m = \frac{1}{3}(2^m - (-1)^m)$ for each $m \leq k$ and we will show $J_{k+1} = \frac{1}{3}(2^{k+1} - (-1)^{k+1})$. Now

$$J_{k+1} = J_k + 2J_{k-1} = \frac{1}{3}(2^k - (-1)^k) + 2 \cdot \frac{1}{3}(2^{k-1} - (-1)^{k-1}) =$$

$$\frac{1}{3}(2^k - (-1)^k + 2 \cdot 2^{k-1} - 2 \cdot (-1)^{k-1}) = \frac{1}{3}(2^k + 2 \cdot 2^{k-1} - (-1)^k + 2 \cdot (-1)^k) =$$

$$\frac{1}{3}(2^k + 2^k - (1-2)(-1)^k) = \frac{1}{3}(2 \cdot 2^k - (-1) \cdot (-1)^k) = \frac{1}{3}(2^{k+1} - (-1)^{k+1}).$$

$\square$

30. For all $n \in \mathbb{N}$, prove that $J_{2n} = \frac{1}{3}(4^n - 1)$.

31. For all $n \in \mathbb{N}$, prove that $J_{2n+1} = \frac{2}{3}(4^n - 1) + 1$.

32. For all natural numbers greater than two, $5J_n = J_{n+2} + 4J_{n-2}$.

    Proof: When $n = 3$ we get
    $$5J_3 = 5 \cdot 3 = 15 = 11 + 4 \cdot 1 = J_5 + 4J_1$$
    and when $n = 4$ we have
    $$5J_4 = 5 \cdot 5 = 25 = 21 + 4 \cdot 1 = J_6 + 4J_2.$$
    This completes our basis cases.

    Next assume that $5J_m = J_{m+2} + 4J_{m-2}$ for all $m \leq k$, and we will show that $5J_{k+1} = J_{k+3} + 4J_{k-1}$. Using strong induction, we have
    $$5J_{k+1} = 5(J_k + 2J_{k-1}) = 5J_k + 2(5J_{k-1}) = J_{k+2} + 4J_{k-2} + 2(J_{k+1} + 4J_{k-3}) =$$
    $$J_{k+2} + 2J_{k+1} + 4J_{k-2} + 8J_{k-3} = J_{k+2} + 2J_{k+1} + 4(J_{k-2} + 2J_{k-3}) = J_{k+3} + 4J_{k-1}.$$
    □

33. For any natural number n, prove that the sum of the first $2n - 1$ Jacobsthal numbers is J_{2k}.

    Proof: We are trying to show $J_1 + J_2 + \cdots + J_{2n-1} = J_{2n}$. For $n = 1$ we get $J_1 = J_2$ which is true. For $n = 2$ we have $J_1 + J_2 + J_3 = J_4$ which is true since $1 + 1 + 3 = 5$.

    Next assume $J_1 + J_2 + \cdots + J_{2k-1} = J_{2k}$, and we must show $J_1 + J_2 + \cdots + J_{2k-1} + J_{2k+1} = J_{2k+2}$. We know
    $$(J_1 + J_2 + \cdots + J_{2k-1}) + J_{2k} + J_{2k+1} = J_{2k} + J_{2k} + J_{2k+1} = J_{2k+1} + 2J_{2k} = J_{2k+2}.$$
    □

34. Prove the sum $J_2 + J_3 + \cdots + J_{n-1} + J_n = \frac{1}{2}(J_{n+2} - 3)$ for any $n \in \mathbb{N}$.

35. For any $n \in \mathbb{N}$ with $n \geq 2$ prove that $J_n^2 = J_{n+1}J_{n-1} + (-2)^{n-1}$.

    Proof: For our basis cases, notice that $J_2^2 = J_3J_1 + (-2)^1$ because $1^2 = 3 \cdot 1 + (-2)$, and $J_3^2 = J_4J_2 + (-2)^2$ because $3^2 = 5 \cdot 1 + 4$.

    Next assume that $J_m^2 = J_{m+1}J_{m-1} + (-2)^{m-1}$ for all $m \leq k$ and we must show $J_{k+1}^2 = J_{k+2}J_k + (-2)^k$. We apply the inductive hypothesis only once and only to J_{k-1}^2 when we take
    $$J_{k+1}^2 = (J_k + 2J_{k-1})^2 = J_k^2 + 4J_kJ_{k-1} + 4J_{k-1}^2 = J_k^2 + 4J_kJ_{k-1} + 4(J_kJ_{k-2} + 2^{k-2}) =$$
    $$J_k^2 + 4J_kJ_{k-1} + 4J_kJ_{k-2} + 2^2 \cdot 2^{k-2} = (J_k^2 + 2J_kJ_{k-1}) + (2J_kJ_{k-1} + 4J_kJ_{k-2}) + 2^k =$$
    $$J_k(J_k + 2J_{k-1}) + 2J_k(J_{k-1} + 2J_{k-2}) + 2^k = J_k(J_{k+1}) + 2J_k(J_k) + 2^k =$$
    $$J_k(J_{k+1} + 2J_k) + 2^k = J_k(J_{k+2}) + 2^k.$$
    □

36. Prove that all the Jacobsthal numbers are odd.

37. For any $n \in \mathbb{N}$ prove that the number J_{3n} is divisible by three.

38. For any $n \in \mathbb{N}$ prove that the number J_{4n} is divisible by five.

    Proof: For our basis step notice that $J_4 = 5$ which is divisible by five.

    Next assume that five divides J_{4k} and thus $5r = J_{4k}$ and we will show that five divides $J_{4(k+1)}$. Now

$$J_{4(k+1)} = J_{4k+4} = J_{4k+3} + 2J_{4k+2} = (J_{4k+2} + 2J_{4k+1}) + 2J_{4k+2} = 2J_{4k+1} + 3J_{4k+2} =$$

$$2J_{4k+1} + 3(J_{4k+1} + 2J_{4k}) = 5J_{4k+1} + 6J_{4k} = 5J_{4k+1} + 6 \cdot 5r = 5(J_{4k+1} + 6r).$$

    Setting s to be the integer $J_{4k+1} + 6r$ shows that five divides $J_{4(k+1)}$. □

39. For any $n \in \mathbb{N}$ prove that the number J_{5n} is divisible by eleven.

40. Find the flaw in the following incorrect proof that all Jacobsthal numbers are divisble by three.

    Incorrect proof: Assume that J_m is divisible by three for all $m \leq k$ and we will show that J_{k+1} is divisible by three. $J_{k+1} = J_k + 2J_{k-1}$. We know J_k and J_{k-1} are divisible by three so $J_k = 3r$ and $J_{k-1} = 3s$ for some $r, s \in \mathbb{Z}$. Thus $J_{k+1} = J_k + 2J_{k-1} = 3r + 2(3s) = 3(r + 2s)$. Set t to be the integer $r + 2s$ to get $J_{k+1} = 3t$ and thus show J_{k+1} is divisible by three. □

Chapter 4

Proofs with Sets and Set Operations

4.1 Proofs with Subsets and Equality

Prove the following statements are true for all sets A, B and C.

1. $\emptyset \subseteq A$

   Proof: We must show if $x \in \emptyset$ then $x \in A$ but the first part is false so the conditional is true. $\square$

2. $A \subseteq A$

   Proof: We must show if $x \in A$ then $x \in A$. Since $P \Rightarrow P$ is a tautology this is true. $\square$

3. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

   Proof: Assume $A \subseteq B$ and $B \subseteq C$ and assume $z \in A$. Then since $z \in A$ and $x \in A \Rightarrow x \in B$, we know $z \in B$. Since $z \in B$ and $x \in b \Rightarrow x \in C$, we know $z \in C$. $\square$

4. Any set with no elements is equal to the empty set.

   Proof: Let A be a set with no elements. We must show both $A \subset \emptyset$ and $\emptyset \subset A$. This means we have to show $x \in A \Rightarrow x \in \emptyset$ and $x \in \emptyset \Rightarrow x \in A$. The first part of both conditional statements is false, so both are true. $\square$

5. Prove that if $A \subseteq B$ and $A \neq \emptyset$ then $B \neq \emptyset$.

   Proof: Assume $A \subseteq B$ and $A \neq \emptyset$. Since A is not empty there is some element $z \in A$. Since $x \in A \Rightarrow x \in B$ and $z \in A$ we know $z \in B$. Since $z \in B$ we know $B \neq \emptyset$. $\square$

   Proof: (Alternate Proof by Contradiction) Assume $A \subseteq B$ and $A \neq \emptyset$ and $B = \emptyset$. Since $A \neq \emptyset$ we know there is some $z \in A$. Since $A \subseteq B$ we know $z \in B$, a contradiction. $\square$

6. Prove that if $A \subseteq \emptyset$ then $A = \emptyset$.

7. Prove that if $A \subseteq B$ and $x \notin B$ then $x \notin A$.

8. Prove that if $A \subseteq B$ and $B \subseteq C$ and $C \subseteq A$ then $A = B$.

9. Prove that if $A \subseteq B$ and $B \subseteq C$ and $C \subseteq A$ then $A = B$ and $B = C$ and $A = C$.

   Proof: We need to show $B \subseteq A$, $C \subseteq B$ and $A \subseteq C$.

   If $x \in B$ then as $B \subseteq C$ we know $x \in C$. Then as $x \in C$ and $C \subset A$ we know $x \in A$. this shows $B \subseteq A$.

   If $x \in C$ then as $C \subseteq A$ we know $x \in A$. Then as $x \in A$ and $A \subset B$ we know $x \in B$. this shows $C \subseteq B$.

   If $x \in A$ then as $A \subseteq B$ we know $x \in B$. Then as $x \in B$ and $B \subset C$ we know $x \in C$. this shows $A \subseteq C$. $\square$

10. Prove that if $A \subseteq B$ and $C \nsubseteq B$ then $C \nsubseteq A$.

    Proof: Suppose that $A \subseteq B$ and $C \nsubseteq B$, yet $C \subseteq A$. We know $C \neq \emptyset$, otherwise C would be a subset of B. As $C \nsubseteq B$ we know C must be nonempty and there must be some element x in C so that $x \notin B$. As $x \in C$ and $C \subseteq A$ we know $x \in A$. As $x \in A$ and $A \subseteq B$ we know $x \in B$ which is a contradiction. $\square$

4.2 Proofs with Power Sets and Cartesian Products

Prove the following statements are true for all sets A, B, C and D.

1. $A \in \mathcal{P}(A)$.

2. $\emptyset \in \mathcal{P}(A)$.

   Proof: The statement $x \in \emptyset$ implies $x \in A$ is automatically true, as the first part is always false. Therefore $\emptyset \subseteq A$ which is what we needed to show. $\square$

3. If $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

   Proof: Suppose $x \in \mathcal{P}(A)$, then $x \subseteq A$. Since $x \subseteq A$ and $A \subseteq B$, we know (by transitivity of $\subseteq$) that $x \subseteq B$. This means $x \in \mathcal{P}(B)$. $\square$

4. If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ then $A \subseteq B$.

   Proof: Suppose $x \in A$. Then $\{x\} \subseteq A$ so $\{x\} \in \mathcal{P}(A)$. Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we know $\{x\} \in \mathcal{P}(B)$. This means $\{x\} \subseteq B$ so $x \in B$. $\square$

5. $A = B$ if and only if $\mathcal{P}(A) = \mathcal{P}(B)$.

6. If $A \in \mathcal{P}(B)$ and $B \in \mathcal{P}(C)$ then $A \in \mathcal{P}(C)$

7. $A \times \emptyset = \emptyset$.

8. If $A \times B \neq \emptyset$ then $A \neq \emptyset$.

9. $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.

   Proof: First suppose $A = \emptyset$ or $B = \emptyset$. If $(a, b) \in A \times B$ then $x \in A$ and $x \in B$, which contradicts one of our two assumptions.

   For the other direction we take the contrapositive and prove "If $A \neq \emptyset$ and $B \neq \emptyset$ then $A \times B \neq \emptyset$." Assume $A \neq \emptyset$ and $B \neq \emptyset$, so there is some $a \in A$ and some $b \in B$. Then $(a, b) \in A \times B$ which proves $A \times B$ is nonempty. $\square$

10. If $A \subseteq B$ then $A \times C \subseteq B \times C$.

11. If $B \subseteq C$ then $A \times B \subseteq A \times C$.

    Proof: If $(x, y) \in A \times B$ then $x \in A$ and $y \in B$. As $y \in B$ and $B \subseteq C$ we know $y \in C$. We now know $x \in A$ and $y \in C$ which means $(x, y) \in A \times C$. $\square$

12. If $A \subseteq B$ and $A \subseteq C$ then $A \times A \subseteq B \times C$.

13. If $A \subseteq C$ and $B \subseteq C$ then $A \times B \subseteq C \times C$.

14. If $C \neq \emptyset$ and $A \times C \subseteq B \times C$ then $A \subseteq B$.

15. If $A = B$ then $A \times C = B \times C$.

16. If $C \neq \emptyset$ and $A \times C = B \times C$ then $A = B$.

    Proof: As $C \neq \emptyset$ we know there is some $y \in C$. We then know

    $x \in A$ iff

    $(x, y) \in A \times C$ iff

    $(x, y) \in B \times C$ iff

    $x \in B$. $\square$

17. If $A \subseteq C$ and $B \subseteq D$ then $A \times B \subseteq C \times D$.

    Proof: Suppose $A \subseteq C$ and $B \subseteq D$ and $(x, y) \in A \times B$. Then we know $x \in A$ and $y \in B$. Since $x \in A$ and $A \subseteq C$, we know $x \in C$. Since $y \in B$ and $B \subseteq D$, we know $y \in D$. As $x \in C$ and $y \in D$ we know $(x, y) \in C \times D$. $\square$

18. If $B \neq \emptyset$ and $A \times B \subseteq C \times D$ then $A \subseteq C$.

    Proof: To show $A \subseteq C$ we first assume $x \in A$. Since $B \neq \emptyset$ we know there is some $y \in B$. As $x \in A$ and $y \in B$ we know $(x, y) \in A \times B$. Since $(x, y) \in A \times B$ and $A \times B \subseteq C \times D$ we know $(x, y) \in C \times D$. Finally since $(x, y) \in C \times D$ we know $x \in C$. $\square$

19. If $A \neq \emptyset$, $B \neq \emptyset$, and $A \times B \subseteq C \times D$ then $A \subseteq C$ and $B \subseteq D$.

20. If $A \times D \subseteq B \times D$, $B \times D \subset C \times D$ and $D \neq \emptyset$ then $A \subseteq C$.

4.3 Proofs with Unions and Intersections

Prove the following statements for any sets A, B, C and D.

1. $A \cup A = A$.
2. $A \cap A = A$.

   Proof: If $x \in A \cap A$ then $x \in A$ and $x \in A$. Thus $x \in A$ so $A \cap A \subseteq A$.

   If $x \in A$ then ($x \in A$ and $x \in A$) so $x \in A \cap A$. This shows $A \subseteq A \cap A$. $\square$

   Proof: (Alternate Style) $x \in A$ iff

   $x \in A$ and $x \in A$. iff

   $x \in A \cap A$. $\square$
3. $A \cap \emptyset = \emptyset$.

   Proof: If $x \in A \cap \emptyset$ then $x \in A$ and $x \in \emptyset$. But $x \in \emptyset$ is a contradiction. $\square$
4. $A \cup \emptyset = A$.
5. $A \cap B \subseteq A$.
6. $A \subseteq A \cup B$.

   Proof: Suppose $x \in A$. Then $x \in A$ or $x \in B$ so $x \in A \cup B$. $\square$
7. Prove $A \cap B \subseteq A \cup B$.
8. $A \subseteq B$ iff $A \cap B = A$.
9. $A \subseteq B$ iff $A \cup B = B$.

   Proof: We will first show if $A \subseteq B$ then $A \cup B = B$, and then show that if $A \cup B = B$ then $A \subseteq B$. Also, to show $A \cup B = B$ we will show each is a subset of the other, so there are actually three things we need to prove.

   Suppose $A \subseteq B$. If $x \in A \cup B$ then $x \in A$ or $x \in B$. If $x \in A$ then $x \in B$ since $A \subseteq B$, so either way $x \in B$. Also if $x \in B$ then $x \in A$ or $x \in B$ so $x \in A \cup B$.

   Now suppose $A \cup B = B$. Assume $x \in A$. Therefore $x \in A$ or $x \in B$ so $x \in A \cup B$. Since $A \cup B = B$ and $x \in A \cup B$ we know $x \in B$. $\square$
10. $A \cap (B \cap C) = (A \cap B) \cap C$.
11. $A \cup (B \cup C) = (A \cup B) \cup C$.

    Proof: $x \in A \cup (B \cup C)$ iff

    $x \in A$ or $x \in B \cup C$ iff

    $x \in A$ or $x \in B$ or $x \in C$ iff

    $x \in A \cup B$ or $x \in C$ iff

    $x \in (A \cup B) \cup C$. $\square$

12. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

13. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

    Proof: $x \in A \cup (B \cap C)$ iff

    $x \in A$ or $x \in B \cap C$ iff

    $x \in A$ or ($x \in B$ and $x \in C$) iff

    ($x \in A$ or B) and ($x \in A$ or C) iff

    $x \in A \cup B$ and $x \in A \cup C$ iff

    $x \in (A \cup B) \cap (A \cup C)$. □

14. If $B \subseteq C$ then $A \cap B \subseteq A \cap C$.

    Proof: Suppose $B \subseteq C$ and $x \in A \cap B$. Then $x \in A$ and $x \in B$. As $x \in B$ and $B \subseteq C$ we know $x \in C$, therefore we know $x \in A$ and $x \in C$. This means $x \in A \cap C$. □

15. If $B \subseteq C$ then $A \cup B \subseteq A \cup C$.

16. If $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$.

    Proof: Assume $A \subseteq C$ and $B \subseteq C$ and that $x \in A \cup B$. We know $x \in A$ or $x \in B$. If $x \in A$ then since $A \subseteq C$ we know $x \in C$. If $x \in B$ then since $B \subseteq C$ we also know $x \in C$. □

17. If $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.

18. $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.

19. $(A \cup B) \cap C \subseteq (A \cap C) \cup B$.

    Proof: Suppose $x \in (A \cup B) \cap C$. Then $x \in A \cup B$ and $x \in C$. Since $x \in A \cup B$, we know $x \in A$ or $x \in B$.

    Case 1: $x \in A$. Since $x \in A$ and $x \in C$ we know $x \in A \cap C$ and thus $x \in A \cap C$ or $x \in B$. This means $x \in (A \cap C) \cup B$.

    Case 2: $x \in B$. Since $x \in B$ we know $x \in A \cap C$ or $x \in B$ so $x \in (A \cap C) \cup B$. □

20. $A \subseteq B \cap C$ iff $A \subseteq B$ and $A \subseteq C$.

21. $A \cup B \subseteq C$ iff $A \subseteq C$ and $B \subseteq C$.

    Proof: ($A \cup B \subseteq C \Rightarrow A \subseteq C$ and $B \subseteq C$.) Suppose $A \cup B \subseteq C$. If $x \in A$ then $x \in A$ or $x \in B$ so $x \in A \cup B$. This is a subset of C so $x \in C$. If $x \in B$ then $x \in A$ or $x \in B$ so we also get $x \in A \cup B$. Again, since $A \cup B \subset C$ we get $x \in C$.

    ($A \subseteq C$ and $B \subseteq C \Rightarrow A \cup B \subseteq C$.) Assume $A \subseteq C$ and $B \subseteq C$ and suppose $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$ then as $A \subseteq C$ we have $x \in C$. If $x \in B$ then since $B \subseteq C$ we have $x \in C$. Either way $x \in C$. □

22. If $A \subseteq B \cup C$ and $A \cap B = \emptyset$, then $A \subseteq C$.

23. If $A \subseteq C$ and $B \subseteq D$ then $A \cup B \subseteq C \cup D$.

24. If $A \subseteq C$ and $B \subseteq D$ then $A \cap B \subseteq C \cap D$.

    Proof: Suppose $x \in A \cap B$. We then know $x \in A$ and $x \in B$. Since $x \in A$ we know $x \in C$. Since $x \in B$ we know $x \in D$. As $x \in C$ and $x \in D$ we have $x \in C \cap D$. $\square$

25. If $A \subseteq C$ and $B \subseteq D$ then $A \cap B \subseteq C \cup D$.

    Proof: Suppose $x \in A \cap B$. We then know $x \in A$ and $x \in B$. Since $x \in A$ we know $x \in C$. As $x \in C$ we know $x \in C$ or $x \in D$ and thus $x \in C \cup D$. $\square$

4.4 Proofs with Setminus and Compliments

Prove the following statements for any sets A, B, C and D. For problems involving complements assume that all sets and elements are contained in some universal set U.

1. $A - \emptyset \subseteq A$.

   Proof: Suppose $x \in A - \emptyset$. Then $x \in A$ and $x \notin \emptyset$ so $x \in A$. $\square$

2. $A \subseteq A - \emptyset$.

   Proof: Suppose $x \in A$. Then $x \in A$ and $x \notin \emptyset$ so $x \in A - \emptyset$. $\square$

3. $(A - B) - B = A - B$.

4. $A - (A - B) = A \cap B$.

5. $(A - B) \cap B = \emptyset$.

6. $(A - B) \cap (A \cap B) = \emptyset$.

   Proof: Suppose $x \in (A-B)\cap(A\cap B)$. Thus $x \in A-B$ meaning $x \in A$ and $x \notin B$. But x is also in $A\cap B$ so x is in A and $x \in B$. Since x can't both be in B and not in B we have reached a contradiction. $\square$

7. $A - B \subseteq (A \cup B) - (A \cap B)$.

   Proof: Suppose $x \in A - B$. Thus we know $x \in A$ and x is not in B. As $x \in A$ we know $x \in A$ or $x \in B$ thus $x \in A \cup B$. Since $x \notin B$ we know x is not in both A and B, so $x \notin A \cap B$. As $x \in A \cup B$ and $x \notin A \cap B$ we know $x \in (A \cup B) - (A \cap B)$. $\square$

8. $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

9. $(A - B) \cup (B - A) \cup (A \cap B) = A \cup B$.

   Proof: Assume $x \in (A - B) \cup (B - A) \cup (A \cap B)$ which means $x \in A - B$ or $x \in B - A$ or $x \in A \cap B$. In the first case $x \in A$ so we know $x \in A$ or $x \in B$ and then $x \in A \cup B$. In the other two cases $x \in B$ so x is in A or B and thus $x \in A \cup B$.

   Next assume $x \in A \cup B$ so $x \in A$ or $x \in B$. If x is in both then $x \in A \cap B$ so we have shown $x \in (A - B) \cup (B - A) \cup (A \cap B)$. If $x \in A$ and not B then we know $x \in A - B$ so we are also done. Finally if $x \in B$ and not A we know $x \in B - A$ so we are done as well. $\square$

10. $(A \cap B) - C \subseteq A - C$.

    Proof: Suppose that $x \in (A \cap B) - C$. Thus we know $x \in A \cap B$ and x is not in C. As $x \in A \cap B$ we know $x \in A$ and $x \in B$ so $x \in A$. We now know $x \in A$ and $x \notin C$ so $x \in A - C$. $\square$

11. $A - (B \cup C) \subseteq (A - B)$.

12. If $A \subseteq B$ then $A - C \subseteq B - C$.

13. If $B \subseteq C$ then $A - C \subseteq A - B$.

    Proof: Suppose $B \subseteq C$ and $x \in A - C$. We know then that $x \in A$ and $x \notin C$. If $x \in B$ then as $B \subset C$ we know $x \in C$, which is a contradiction. Thus $x \notin B$. As $x \in A$ and $x \notin B$ we know $x \in A - B$. $\square$

14. $(A-B)\cap C=(A\cap C)-B$.

15. $(A-B)-C=(A-C)-B$.

    Proof: $x\in(A-B)-C$ iff

    $x\in A-B$ and $x\notin C$ iff

    $x\in A$ and $x\notin B$ and $x\notin C$ iff

    $x\in A$ and $x\notin C$ and $x\notin B$ iff

    $x\in A-C$ and $x\notin B$ iff

    $x\in(A-C)-B$. □

16. $(A-B)-C=(A-C)-(B-C)$.

17. $A-(B\cap C)=(A-B)\cup(A-C)$.

18. $A-(B\cup C)=(A-B)\cap(A-C)$.

    Proof: Suppose $x\in A-(B\cup C)$. Then we know $x\in A$ and x is not in B or C and thus by DeMorgan's Laws, x is not in B and x is not in C. As $x\in A$ and $x\notin B$ we know $x\in A-B$. As $x\in A$ and $x\notin C$ we know $x\in A-C$. Thus $x\in(A-B)\cap(A-C)$.

    Now suppose $x\in(A-B)\cap(A-C)$. Thus x is in both $A-B$ and $A-C$ so $x\in A$, $x\notin B$, $x\in A$, and $x\notin C$. Since $x\notin B$ and $x\notin C$ we know by DeMorgan that x is not in B or C. This means $x\notin B\cup C$. Since $x\in A$ and $x\notin B\cup C$ we know that $x\in A-(B\cup C)$. □

    Proof: (Alternate Proof)

    $x\in A-(B\cup C)$ iff

    $x\in A$ and $x\notin B\cup C$ iff

    $x\in A$ and $x\notin B$ or C iff

    $x\in A$ and $x\notin B$ and $x\notin C$ iff

    $x\in A$ and $x\notin B$ and $x\in A$ and $x\notin C$ iff

    $x\in A-B$ and $x\in A-C$.

    $x\in(A-B)\cap(A-C)$. □

19. $(A\cup B)-C=(A-C)\cup(B-C)$.

20. $(A\cap B)-C=(A-C)\cap(B-C)$.

    Proof: Suppose $x\in(A\cap B)-C$. Then $x\in A\cap B$ and $x\notin C$, which means $x\in A$, $x\in B$ and $x\notin C$. This means $x\in A$ and $x\notin C$ so $x\in A-C$. It also means $x\in B$ and $x\notin C$ so $x\in B-C$. Since $x\in A-C$ and $x\in B-C$ we know $x\in(A-C)\cap(B-C)$.

    Now suppose $x\in(A-C)\cap(B-C)$. Thus we know $x\in A-C$ and $x\in B-C$. This means $x\in A$, $x\notin C$, $x\in B$ and $x\notin C$. This tells us $x\in A\cap B$ and $x\notin C$ so $x\in(A\cap B)-C$. □

    Proof: (Alternate Proof) $x\in(A\cap B)-C$ iff

    $x\in(A\cap B)$ and $x\notin C$ iff

    $x\in A$ and $x\in B$ and $x\notin C$ iff

($x \in A$ and $x \notin C$) and ($x \in B$ and $x \notin C$) iff

$x \in A - C$ and $x \in B - C$ iff

$x \in (A - C) \cap (B - C)$. □

21. If $A \subseteq B$ and $C \subseteq D$ then $C - B \subseteq D - A$.

    Proof: Assume $A \subseteq B$, $C \subseteq D$ and $x \in C - B$. Then $x \in C$ and $x \notin B$. Since $x \in C$ and $C \subseteq D$ we know $x \in D$. Since $A \subseteq B$ and $x \notin B$ we know $x \notin A$. Thus we have shown $x \in D$ and $x \notin A$ which means $x \in D - A$. □

22. $(A^C)^C = A$

    Proof: $x \in (A^C)^C$ iff

    $x \notin A^C$ iff

    $x \in A$. □

23. $A \cap A^C = \emptyset$

    Proof: Suppose $x \in A \cap A^C$. Then $x \in A$ and $x \in A^C$, so $x \in A$ and $x \notin A$, a contradiction. □

24. $A \cup A^C = U$.

    Proof: If $x \in A \cup A^C$ then we know $x \in U$ because we are assuming all x are in contained in our universe.

    For the other direction, if x is any element in our universe, x is either in A or x is not in A. Thus $x \in A \cup A^C$.

25. $\emptyset^C = U$.

    Proof: If $x \in U$ then x can't be in the emptyset because nothing can. Thus x is in $\emptyset^C$.

    If $x \in \emptyset^C$ then $x \in U$ simply because we assumed all elements are contained in our universe. □

26. $U^C = \emptyset$.

27. $(A \cup B)^C = A^C \cap B^C$.

    Proof: $x \in (A \cup B)^C$ iff

    $x \notin (A \cup B)$ iff

    $x \notin (A$ and $B)$ iff

    $x \notin A$ or $x \notin B$ iff

    $x \subseteq A^C$ or $x \subseteq B^C$. □

28. $(A \cap B)^C = A^C \cup B^C$.

29. $(A - B)^C = A^C \cup B$.

30. $A - B = A \cap B^C$.

31. $A - B = B^C - A^C$.

    Proof: $x \in B^C - A^C$ iff

    $x \in B^C$ and $x \notin A^C$ iff

    $x \notin B$ and $x \in A$ iff

    $x \in A$ and $x \notin B$ iff

    $x \in A - B$. $\square$

4.5 Proofs with Combinations of Set Operations

These problems not only involve Power Sets and Cartesian Products but also mix them together with all other set operations from these sheets. Prove the following statements are true for all sets A, B, C and D. For the problems with complements assume that A, B, C, and D are all contained in some universal set U and that products of these sets are contained in $U \times U$.

1. $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.
2. $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.
3. $\mathcal{P}(A - B) \subseteq \mathcal{P}(A)$.

   Proof: Suppose $x \in \mathcal{P}(A-B)$. Then $x \subseteq A-B$. As $A-B \subseteq A$ we know $x \subseteq A$ and thus $x \in \mathcal{P}(A)$. $\square$
4. If $A \cap B = \emptyset$ then $\mathcal{P}(A) \cap \mathcal{P}(B) = \{\emptyset\}$.

   Proof: To show $\{\emptyset\} \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$, just assume $x \in \{\emptyset\}$. Then $x = \emptyset$ so $x \subseteq A$ and $x \subseteq B$. Then $x \in \mathcal{P}(A)$ and $x \in \mathcal{P}(B)$ so $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

   Now suppose $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Since $x \in \mathcal{P}(A)$ and $x \in \mathcal{P}(B)$ we know $x \subseteq A$ and $x \subseteq B$. Now suppose $y \in x$. Since $y \in x$ and $x \subseteq A$ we know $y \in A$. Similarly, we know $y \in x$ and $x \subseteq B$ so $y \in B$. Thus $y \in A \cap B$ which is a contradiction. This means x has no elements . Thus $x = \emptyset$ so $x \in \{\emptyset\}$, which completes the proof. $\square$
5. $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
6. $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
7. $A \times (B - C) = (A \times B) - (A \times C)$.
8. If $A \neq \emptyset$ then $A \times B \subseteq A \times C$ iff $B \subseteq C$.
9. If $B \neq \emptyset$ and $A \times B \subseteq B \times C$ then $A \subseteq C$.

   Proof: Suppose $B \neq \emptyset$, $A \times B \subseteq C \times B$, and $x \in A$. Since B is not empty it contains some elements. Pick one and call it b. Since $x \in A$ and $b \in B$ we know $(x, b) \in A \times B$. As $A \times B \subseteq B \times C$ we know $(x, b) \in C \times B$ which implies $x \in C$ and $b \in B$. As we have shown $x \in C$ we are done. $\square$
10. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
11. $(A \times C) \cup (B \times D) \subseteq (A \cup B) \times (C \cup D)$.

    Proof: Suppose $(x, y) \in (A \times C) \cup (B \times D)$. Then $(x, y) \in A \times C$ or $(x, y) \in B \times D$. Thus ($x \in A$ and $y \in C$) or ($x \in B$ and $y \in D$). We then know $x \in A$ or $x \in B$. Similarly $y \in C$ or $y \in D$. Thus $x \in A \cup B$ and $y \in C \cup D$. This means $(x, y) \in (A \cup B) \times (C \cup D)$. $\square$
12. $(A \times C) \cap (B \times D) \subseteq (A \cup B) \times (C \cup D)$.
13. $(A \cap B) \times (C \cup D) \subseteq (A \times C) \cup (B \times D)$.
14. $(A \times C) \cup (B \times D) = (A - B) \times C \cup (A \cap B) \times (C \cup D) \cup (B - A) \times D$.
15. $A^C \times B^C \subseteq (A \times B)^C$.

16. $A^C \times B \subseteq (A \times B)^C$.

17. $(A \times B)^C = (A^C \times B^C) \cup (A^C \times B) \cup (A \times B^C)$.

    Proof: Suppose first that $(x, y) \in (A \times B)^C$, thus it is not the case that $x \in A$ and $y \in B$. Thus either ($x \notin A$ and $y \notin B$) or ($x \notin A$ and $y \in B$) or ($x \in A$ and $y \notin B$.) This means either $(x, y) \in (A^C \times B^C)$ or $(x, y) \in (A^C \times B)$ or $(x, y) \in (A \times B^C)$. Therefore $(x, y) \in (A^C \times B^C) \cup (A^C \times B) \cup (A \times B^C)$.

    Now suppose that $(x, y) \in (A^C \times B^C) \cup (A^C \times B) \cup (A \times B^C)$, so we know that $(x, y) \in (A^C \times B^C)$ or $(x, y) \in (A^C \times B)$ or $(x, y) \in (A \times B^C)$.

    Case 1: $(x, y) \in (A^C \times B^C)$. Here $x \in A^C$ so $x \notin A$ so $(x, y) \notin A \times B$. Thus $(x, y) \in (A \times B)^C$.

    Case 2: $(x, y) \in (A^C \times B)$. Here $x \in A^C$ so we arrive at the same conclusion as case 1.

    Case 3: $(x, y) \in (A \times B^C)$. Here $y \in B^C$ so $y \notin B$ so $(x, y) \notin A \times B$. Thus $(x, y) \in (A \times B)^C$. □

Disprove the following false statements.

1. $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.

   Disproof: If $A = \{1\}$ and $B = \{2\}$ then $\mathcal{P}(A \cup B) = \mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ but $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}\} \cup \{\emptyset, \{2\}\} = \{\emptyset, \{1\}\, \{2\}\}$. As $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\} \not\subseteq \{\emptyset, \{1\}\, \{2\}\}$, we have found a counterexample.

2. $\mathcal{P}(A - B) \subseteq \mathcal{P}(A) - \mathcal{P}(B)$.

   Disproof: If $A = \{1\}$ and $B = \{2\}$ then $\mathcal{P}(A - B) = \mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$ but $\mathcal{P}(A) - \mathcal{P}(B) = \{\emptyset, \{1\}\} - \{\emptyset, \{2\}\} = \{\{1\}\}$. As $\{\emptyset, \{1\}\} \not\subseteq \{\{1\}\}$, we have found a counterexample.

3. $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$.

   Disproof: If $A = \{1, 2\}$ and $B = \{2\}$ then $\mathcal{P}(A) - \mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} - \{\emptyset, \{2\}\} = \{\{1\}, \{1, 2\}\}$ but $\mathcal{P}(A - B) = \mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$. As $\{\{1\}, \{1, 2\}\} \not\subseteq \{\emptyset, \{1\}\}$, we have found a counterexample.

4. $(A \times B)^C \subseteq A^C \times B^C$.

   Disproof: If $U = \{1, 2\}$, $A = \{1\}$, and $B = \{2\}$ then $A \times B = \{(1, 2)\}$ so $(A \times B)^C = \{(1, 1), (2, 2), (2, 1)\}$. However, $A^C = B$ and $B^C = A$ so $A^C \times B^C = \{(2, 1)\}$. Since $\{(1, 1), (2, 2), (2, 1)\} \not\subseteq \{(2, 1)\}$, we have found a counterexample.

5. $(A \times B)^C \subseteq A^C \times B$.

Chapter 5

Relations

5.1 Relations

1. Write out the elements of the following relations on $A = \{1, 2, 3, 4\}$ and state the cardinality of R. Find $\text{Dom}(R)$ and $\text{Ran}(R)$.

   (a) $R = \{(x, x) \in A \times A\}$. [Answer: $R = \{(1, 1),\ (2, 2),\ (3, 3),\ (4, 4)\ \}$. $|R| = 4$. $\text{Dom}(R) = A$ and $\text{Ran}(R) = A$.]

   (b) $R = \{(x, y) \in A \times A : x = 1\}$. [Answer: $R = \{(1, 2),\ (1, 2),\ (1, 3),\ (1, 4)\ \}$. $|R| = 4$. $\text{Dom}(R) = \{1\}$ and $\text{Ran}(R) = A$.]

   (c) $R = \{(x, y) \in A \times A : x < y\}$. [Answer: $R = \{(1, 2),\ (1, 3),\ (1, 4),\ (2, 3),\ (2, 4),\ (3, 4)\}$. $|R| = 6$. $\text{Dom}(R) = \{1, 2, 3\}$ and $\text{Ran}(R) = \{2, 3, 4\}$.]

   (d) $R = \{(x, y) \in A \times A : x \neq y\}$. [Answer: $R = \{(1, 2),\ (1, 3),\ (1, 4),\ (2, 1),\ (2, 3),\ (2, 4),\ (3, 1),\ (3, 2),\ (3, 4),\ (4, 1),\ (4, 2),\ (4, 3)\ \}$. $|R| = 12$. $\text{Dom}(R) = A$ and $\text{Ran}(R) = A$.]

   (e) $R = \{(x, y) \in A \times A : x \leq y\}$. [Answer: $R = \{(1, 1),\ (1, 2),\ (1, 3),\ (1, 4),\ (2, 2),\ (2, 3),\ (2, 4),\ (3, 3),\ (3, 4),\ (4, 4)\}$. $|R| = 10$. $\text{Dom}(R) = A$ and $\text{Ran}(R) = A$.]

   (f) $R = \{(x, y) \in A \times A : y - x = 1\}$. [Answer: $R = \{(1, 2),\ (2, 3),\ (3, 4)\}$. $|R| = 3$. $\text{Dom}(R) = \{1, 2, 3\}$ and $\text{Ran}(R) = \{2, 3, 4\}$.]

   (g) $R = \{(x, y) \in A \times A : |y - x| = 1\}$. [Answer: $R = \{(1, 2),\ (2, 1),\ (2, 3),\ (3, 2),\ (4, 3),\ (3, 4)\}$. $|R| = 6$. $\text{Dom}(R) = A$ and $\text{Ran}(R) = A$.]

   (h) $R = \{(x, y) \in A \times A : y - x = 2\}$. [Answer: $R = \{(1, 3),\ (2, 4)\}$. $|R| = 2$. $\text{Dom}(R) = \{1, 2\}$ and $\text{Ran}(R) = \{3, 4\}$.]

   (i) $R = \{(x, y) \in A \times A : |y - x| = 2\}$. [Answer: $R = \{(1, 3), (3, 1), (2, 4), (4, 2)\}$. $|R| = 4$. $\text{Dom}(R) = A$ and $\text{Ran}(R) = A$.]

   (j) $R = \{(x, y) \in A \times A : |y - x| = 4\}$. [Answer: $R = \emptyset$. $|R| = 0$. $\text{Dom}(R) = \emptyset$ and $\text{Ran}(R) = \emptyset$.]

   (k) $R = \{(x, y) \in A \times A : x + y \in \mathbb{E}\}$. [Answer: $R = \{(1, 1),\ (1, 3),\ (2, 2),\ (2, 4),\ (3, 1),\ (3, 3),\ (4, 2), (4, 4)\}$. $|R| = 8$. $\text{Dom}(R) = A$ and $\text{Ran}(R) = A$.]

   (l) $R = \{(x, y) \in A \times A : x + y \notin \mathbb{E}\}$. [Answer: $R = \{(1, 2),\ (2, 1),\ (1, 4),\ (4, 1),\ (2, 3), (3, 2),\ (3, 4), (4, 3)\}$. $|R| = 8$. $\text{Dom}(R) = A$ and $\text{Ran}(R) = A$.]

(m) $R = \{(x,y) \in A \times A : (x \in \mathbb{E}) \wedge (y \notin \mathbb{E})\}$. [Answer: $R = \{(2,1), (2,3), (4,1), (4,3)\}$. $|R| = 4$. $\text{Dom}(R) = \{2,4\}$ and $\text{Ran}(R) = \{1,3\}$.]

(n) $R = \{(x,y) \in A \times A : (x \in \mathbb{E}) \vee (y \notin \mathbb{E})\}$. [Answer: $R = \{(1,1), (1,3), (2,1), (2,2), (2,3), (2,4), (3,1), (3,3), (4,1), (4,2), (4,3), (4,4)\}$. $|R| = 12$. $\text{Dom}(R) = A$ and $\text{Ran}(R) = A$.]

(o) $R = \{(x,y) \in A \times A : (x \in \mathbb{E}) \Leftrightarrow (y \notin \mathbb{E})\}$. [Answer: $R = \{(1,2), (1,4), (2,1), (2,3), (3,2), (3,4), (4,1), (4,3)\}$. $|R| = 8$. $\text{Dom}(R) = A$ and $\text{Ran}(R) = A$.]

(p) $R = \{(x,y) \in A \times A : x+y \text{ is prime.}\}$. [Answer: $R = \{(1,1), (1,2), (2,1), (1,4), (4,1), (2,3), (3,2), (3,4), (4,3)\}$ 9. $\text{Dom}(R) = A$ and $\text{Ran}(R) = A$.]

(q) $R = \{(x,y) \in A \times A : xy \text{ is prime.}\}$. [Answer: $R = \{(1,2), (2,1), (1,3), (3,1)\}$. $|R| = 4$. $\text{Dom}(R) = \{1,2,3\}$ and $\text{Ran}(R) = \{1,2,3\}$.]

(r) $R = \{(x,y) \in A \times A : y - x \text{ is prime.}\}$. [Answer: $R = \{(1,3), (2,4), (1,4)\}$. $|R| = 3$. $\text{Dom}(R) = \{1,2\}$ and $\text{Ran}(R) = \{3,4\}$.]

(s) $R = \{(x,y) \in A \times A : x^y \text{ is prime.}\}$. [Answer: $R = \{(2,1), (3,1)\}$. $|R| = 2$. $\text{Dom}(R) = \{2,3\}$ and $\text{Ran}(R) = \{1\}$.]

(t) $R = \{(x,y) \in A \times A : x \mid y\}$. [Answer: $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$. $|R| = 8$. $\text{Dom}(R) = A$ and $\text{Ran}(R) = A$.]

(u) $R = \{(x,y) \in A \times A : x \mid (y-1)\}$. [Answer: $R = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,3), (3,1), (3,4), (4,1)\}$. $|R| = 9$. $\text{Dom}(R) = A$ and $\text{Ran}(R) = A$.]

(v) $R = \{(x,y) \in A \times A : (x-1) \mid y\}$. [Answer: $R = \{(2,1), (2,2), (2,3), (2,4), (3,2), (3,4), (4,3)\}$. $|R| = 7$. $\text{Dom}(R) = \{2,3,4\}$ and $\text{Ran}(R) = A$.]

(w) $R = \{(x,y) \in A \times A : 5 \mid x+y\}$. [Answer: $R = \{(1,4), (2,3), (3,2), (4,1)\}$. $|R| = 4$. $\text{Dom}(R) = \text{Ran}(R) = A$.]

(x) $R = \{(x,y) \in A \times A : 3 \mid |y-x|\}$. [Answer: $R = \{(1,1), (1,4), (2,2), (3,3), (4,1)\ (4,4)\}$. $|R| = 6$. $\text{Dom}(R) = A$ and $\text{Ran}(R) = A$.]

(y) $R = \{(x,y) \in A \times A : 4 \mid |y-x|\}$. [Answer: $R = \{(1,1), (2,2), (3,3), (4,4)\}$. $|R| = 4$. $\text{Dom}(R) = A$ and $\text{Ran}(R) = A$.]

(z) $R = \{(x,y) \in A \times A : x^y < 10\}$. [Answer: $R = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,2), (4,1)\}$. $|R| = 10$. $\text{Dom}(R) = A$ and $\text{Ran}(R) = A$.]

2. Write out the elements of the following relations from $A = \{1,2,3,4\}$ to $B = \{5,6,7,8\}$ and state the cardinality of R. Find $\text{Dom}(R)$ and $\text{Ran}(R)$.

(a) $R = \{(x,x) \in A \times B\}$. [Answer: $R = \emptyset$. $|R| = 0\}$. $\text{Dom}(R) = \emptyset$ and $\text{Ran}(R) = \emptyset$.]

(b) $R = \{(x,y) \in A \times B : x = 1\}$. [Answer: $R = \{(1,5), (1,6), (1,7), (1,8)\}$ $|R| = 4\}$.$\text{Dom}(R) = \{1\}$ and $\text{Ran}(R) = B$.]

(c) $R = \{(x,y) \in A \times B : x < y\}$. [Answer: $R = A \times B$. $|R| = 16$. $\text{Dom}(R) = A$ and $\text{Ran}(R) = B$.]

(d) $R = \{(x,y) \in A \times B : x \neq y\}$. [Answer: $R = A \times B$. $|R| = 16$. $\text{Dom}(R) = A$ and $\text{Ran}(R) = B$.]

(e) $R = \{(x,y) \in A \times B : x \geq y\}$. [Answer: $R = \emptyset$. $|R| = 0$. $\text{Dom}(R) = \emptyset$ and $\text{Ran}(R) = \emptyset$.]

(f) $R = \{(x,y) \in A \times B : x + y \in \mathbb{E}\}$. [Answer: $R = \{(1,5), (1,7), (2,6), (2,8), (3,5), (3,7), (4,6), (4,8)\}$. $|R| = 8$. $\text{Dom}(R) = A$ and $\text{Ran}(R) = B$.]

(g) $R = \{(x, y) \in A \times B : x + y \notin \mathbb{E}\}$. [Answer: $R = \{(1, 6),\ (2, 5),\ (1, 8),\ (4, 5),\ (2, 7), (3, 6), (3, 8), (4, 7)\}$. $|R| = 8$. $\mathrm{Dom}(R) = A$ and $\mathrm{Ran}(R) = B$.]

(h) $R = \{(x, y) \in A \times B : x + y$ is prime.$\}$. [Answer: $R = \{(1, 6),\ (2, 5),\ (3, 8),\ (4, 7)\}$. $|R| = 4$. $\mathrm{Dom}(R) = A$ and $\mathrm{Ran}(R) = B$.]

(i) $R = \{(x, y) \in A \times B : xy$ is prime.$\}$. [Answer: $R = \{(1, 5),\ (1, 7)\}$. $|R| = 2$. $\mathrm{Dom}(R) = \{1\}$ and $\mathrm{Ran}(R) = \{5, 7\}$.]

(j) $R = \{(x, y) \in A \times B : y - x$ is prime.$\}$. [Answer: $R = \{(1, 6),\ (1, 8),\ (2, 5),\ (2, 7),\ (3, 5),\ (3, 6),\ (3, 8),\ (4, 7)\}$. $|R| = 8$. $\mathrm{Dom}(R) = A$ and $\mathrm{Ran}(R) = B$.]

(k) $R = \{(x, y) \in A \times B : x \mid y\}$. [Answer: $R = \{(1, 5),\ (1, 6),\ (1, 7),\ (1, 8),\ (2, 6),\ (2, 8),\ (3, 6),\ (4, 8), \}$. $|R| = 8$. $\mathrm{Dom}(R) = A$ and $\mathrm{Ran}(R) = B$.]

(l) $R = \{(x, y) \in A \times B : y - x = 1\}$. [Answer: $R = \{(4, 5)\}$. $|R| = 1$. $\mathrm{Dom}(R) = \{4\}$ and $\mathrm{Ran}(R) = \{5\}$.]

(m) $R = \{(x, y) \in A \times B : |y - x| = 1\}$. [Answer: $R = \{(4, 5)\}$. $|R| = 1$. $\mathrm{Dom}(R) = \{4\}$ and $\mathrm{Ran}(R) = \{5\}$.]

(n) $R = \{(x, y) \in A \times B : |y - x| = 2\}$. [Answer: $R = \{(3, 5),\ (4, 6)\}$. $|R| = 2$. $\mathrm{Dom}(R) = \{3, 4\}$ and $\mathrm{Ran}(R) = \{5, 6\}$.]

(o) $R = \{(x, y) \in A \times B : |y - x| = 3\}$. [Answer: $R = \{(2, 5), (3, 6), (4, 7)\}$. $|R| = 3$. $\mathrm{Dom}(R) = \{2, 3, 4\}$ and $\mathrm{Ran}(R) = \{5, 6, 7\}$.]

(p) $R = \{(x, y) \in A \times B : x^y < 10\}$. [Answer: $R = \{(1, 5), (1, 6), (1, 7), (1, 8)\}$. $|R| = 4$. $\mathrm{Dom}(R) = \{1\}$ and $\mathrm{Ran}(R) = B$.]

(q) $R = \{(x, y) \in A \times B : x^y < 100\}$. [Answer: $R = \{(1, 5),\ (1, 6),\ (1, 7),\ (1, 8),\ (2, 5),\ (2, 6)\}$. $|R| = 6$. $\mathrm{Dom}(R) = \{1, 2\}$ and $\mathrm{Ran}(R) = B$.]

(r) $R = \{(x, y) \in A \times B : y^x < 10\}$. [Answer: $R = \{(1, 5), (1, 6), (1, 7), (1, 8)\}$. $|R| = 4$. $\mathrm{Dom}(R) = \{1\}$ and $\mathrm{Ran}(R) = B$.]

(s) $R = \{(x, y) \in A \times B : y^x < 100\}$. [Answer: $R = \{\ (1, 5),\ (1, 6),\ (1, 7),\ (1, 8),\ (2, 5),\ (2, 6),\ (2, 7),\ (2, 8)\}$. $|R| = 8$. $\mathrm{Dom}(R) = \{1, 2\}$ and $\mathrm{Ran}(R) = B$.]

3. The following relations are on the set $A = \mathbb{E}$ of all even numbers. For each relation, write three elements that are in R or explain why it is not possible to do so.

(a) $R = \{(x, y) \in A \times A : x = 1\}$. [Answer: Not possible. As both entries must be even we cannot find any elements that are in R.]

(b) $R = \{(x, y) \in A \times A : x \neq y\}$. [Answer: $(2, 4) \in R$, $(2, 6) \in R$, $(12, 18) \in R$.]

(c) $R = \{(x, y) \in A \times A : x + y \notin \mathbb{E}\}$. [Answer: Not possible. The sum of two evens is always in $\mathbb{E}$.]

(d) $R = \{(x, y) \in A \times A : x + y$ is prime.$\}$. [Answer: $(0, 2) \in R$, $(2, 0) \in R$, $(4, -2) \in R$.]

(e) $R = \{(x, y) \in A \times A : xy$ is prime.$\}$. [Answer: Not possible. The product of two evens is a multiple of four, and no multiples of four are prime.]

(f) $R = \{(x, y) \in A \times A : y - x$ is prime.$\}$. [Answer: $(2, 4) \in R$, $(4, 6) \in R$, $(-6, -4) \in R$.]

(g) $R = \{(x, y) \in A \times A : x \mid y\}$. [Answer: $(2, 4) \in R$, $(2, -4) \in R$, $(4, 248) \in R$.]

(h) $R = \{(x, y) \in A \times A : y - x = 1\}$. [Answer: Not possible. The difference will always be an even number, and thus can never be one.]

(i) $R = \{(x, y) \in A \times A : y - x = 2\}$. [Answer: $(0, 2) \in R$, $(2, 4) \in R$, $(4, 6) \in R$.]

(j) $R = \{(x, y) \in A \times A : x^y < 10\}$. [Answer: $(2, 0) \in R$, $(2, -2) \in R$, $(4, -2) \in R$.]

4. Which of the following relations on $A = \mathbb{N}$ are infinite? If they are finite, state the cardinality of R.

(a) $R = \{(x, y) \in A \times A : x = y\}$. [Answer: Infinite.]

(b) $R = \{(x, y) \in A \times A : x = 1\}$. [Answer: Infinite.]

(c) $R = \{(x, y) \in A \times A : y = x^2\}$. [Answer: Infinite.]

(d) $R = \{(x, y) \in A \times A : y^2 = x^2\}$. [Answer: Infinite.]

(e) $R = \{(x, y) \in A \times A : xy = -1\}$. [Answer: Finite. $|R| = 0$.]

(f) $R = \{(x, y) \in A \times A : xy = 0\}$. [Answer: Finite. $|R| = 0$.]

(g) $R = \{(x, y) \in A \times A : xy = 1\}$. [Answer: Finite. $|R| = 1$.]

(h) $R = \{(x, y) \in A \times A : xy = 2\}$. [Answer: Finite. $|R| = 2$.]

(i) $R = \{(x, y) \in A \times A : xy = p\}$ where p is some fixed prime number. [Answer: Finite. $|R| = 2$.]

(j) $R = \{(x, y) \in A \times A : xy = 4\}$. [Answer: Finite. $|R| = 3$.]

(k) $R = \{(x, y) \in A \times A : xy = s\}$ where s is a prime squared. [Answer: Finite. $|R| = 3$.]

(l) $R = \{(x, y) \in A \times A : xy = 6\}$. [Answer: Finite. $|R| = 4$.]

(m) $R = \{(x, y) \in A \times A : x + y = 1\}$. [Answer: Finite. $|R| = 0$.]

(n) $R = \{(x, y) \in A \times A : x + y = 4\}$. [Answer: Finite. $|R| = 3$.]

(o) $R = \{(x, y) \in A \times A : x + y = n\}$ where n is some fixed natural number. [Answer: Finite. $|R| = n - 1$.]

(p) $R = \{(x, y) \in A \times A : x^2 + y^2 = 1\}$. [Answer: Finite. $|R| = 0$.]

(q) $R = \{(x, y) \in A \times A : x^2 + y^2 = 2\}$. [Answer: Finite. $|R| = 1$.]

(r) $R = \{(x, y) \in A \times A : x^2 + y^2 = 5\}$. [Answer: Finite. $|R| = 2$.]

(s) $R = \{(x, y) \in A \times A : x^2 + y^2 = 25\}$. [Answer: Finite. $|R| = 2$.]

(t) $R = \{(x, y) \in A \times A : x + y \text{ is prime }\}$. [Answer: Infinite.]

(u) $R = \{(x, y) \in A \times A : xy = x\}$. [Answer: Infinite.]

(v) $R = \{(x, y) \in A \times A : x^2 y = x\}$. [Answer: Finite. $|R| = 1$.]

5. Which of the above answers change when $\mathbb{N}$ is replaced with $\mathbb{Z}$? [Answer: $e, f, g, h,\ i, j, k,\ l, m, m,\ n, o, p,\ r, s, v$ all change.]

5.2 Relations and Cardinality

1. How many relations R are there on the set $A = \{1, 2\}$ when:
   (a) There are no restrictions? [Answer: $2^{|A|\cdot|A|} = 16$.]
   (b) $|R| = 0$? [Answer: 1. $R = \emptyset$.]
   (c) $|R| = 1$? [Answer: 4. Here R must be any subset of $A \times A$ of size one. Thus $R = \{(1,1)\}, R = \{(1,2)\}, R = \{(2,1)\}$, and $R = \{(2,2)\}$.]
   (d) $|R| = 2$? [Answer: 6. $R = \{(1,1),(1,2)\}, R = \{(1,1),(2,1)\}, R = \{(1,1),(2,2)\}, R = \{(1,2),(2,1)\}, R = \{(1,2),(2,2)\}$, and $R = \{(2,1),(2,2)\}$.]
   (e) $|R| = 3$? [Answer: 4. $R = \{(1,1),(1,2),(2,1)\}, R = \{(1,1),(1,2),(2,2)\}, R = \{(1,1),(2,1),(2,2)\}$, and $R = \{(1,2),(2,1),(2,2)\}$.]
   (f) $|R| = 4$? [Answer: 1. $R = \{(1,1),(1,2),(2,1),(2,2)\}$.]
   (g) $(1,2) \in R$? [Answer: 8]
   (h) $(1,2) \notin R$? [Answer: 8]
   (i) $(1,1)$ and $(2,2)$ are both in R? [Answer: 4]
   (j) $(1,1) \in R$ and $(2,2) \notin R$. [Answer: 4]

2. How many relations R are there on the set $A = \emptyset$? [Answer: The only subset of $\emptyset \times \emptyset$ is the empty set itself. Thus $\emptyset$ is the only relation and the answer is one.]

3. How many relations R are there on the set $A = \{\emptyset\}$ when:
   (a) There are no restrictions? [Answer: $2^{|A|\cdot|A|} = 2$.]
   (b) $|R| = 0$? [Answer: 1. $R = \emptyset$.]
   (c) $|R| = 1$? [Answer: 1. $R = \{(\emptyset, \emptyset)\}$.]

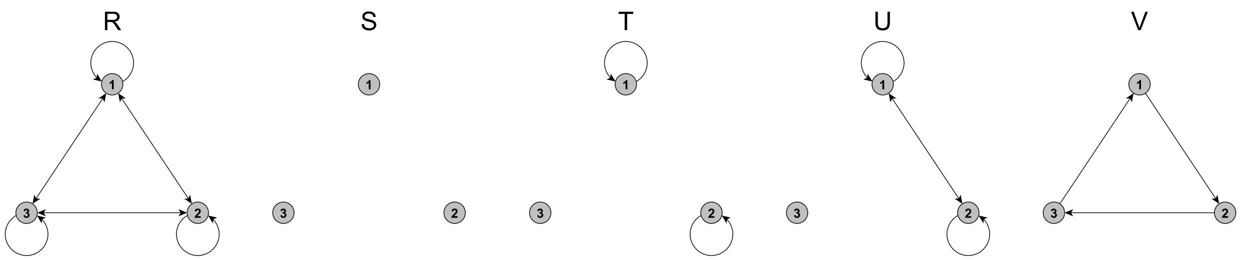
4. How many relations R are there on the set $A = \{1, 2, 3\}$ when:
   (a) There are no restrictions? [Answer: $2^{|A|\cdot|A|} = 512$.]
   (b) $(1,2) \in R$? [Answer: Any element will appear in half of the subsets of $|A| \cdot |A|$ so the answer must be 256]
   (c) $(1,2) \notin R$? [Answer: 256]
   (d) $|R| = 0$? [Answer: 1. $R = \emptyset$.]
   (e) $|R| = 1$? [Answer: 9. Here R must be any subset of $A \times A$ of size one. Thus R can contain any one of nine elements. $R = \{(1,1)\}, R = \{(1,2)\}, R = \{(1,3)\}, R = \{(2,1)\}, R = \{(2,2)\}, R = \{(2,3)\}, R = \{(3,1)\}, R = \{(3,2)\}$ and $R = \{(3,3)\}$.]
   (f) $|R| = 8$? [Answer: 9. These are the relations that have all of $A \times A$ except for one element. There are nine different elements we could skip, so the answer is nine.]
   (g) $|R| = 9$? [Answer: 1. R contains nine elements and is a subset of $A \times A$ which also contains nine elements. Therefore it must be all of $A \times A$.]

5. How many relations R are there from the set $A = \{1, 2\}$ to the set $B = \{3\}$ when:
   (a) There are no restrictions? [Answer: $2^{|A| \cdot |B|} = 4$.]
   (b) $|R| = 0$? [Answer: 1. $R = \emptyset$.]
   (c) $|R| = 1$? [Answer: 2. Here R must be any subset of $A \times B$ of size one. Thus $R = \{(1, 3)\}$, and $R = \{(2, 3)\}$.]
   (d) $|R| = 2$? [Answer: 1. $R = \{(1, 3), (2, 3)\}$.]
   (e) $(1, 3) \in R$? [Answer: 2.]
   (f) $(1, 3) \notin R$? [Answer: 2]
6. How many relations R are there from the set $A = \{1, 2\}$ to the set $B = \{3, 4\}$ when:
   (a) There are no restrictions? [Answer: $2^{|A| \cdot |B|} = 16$.]
   (b) $|R| = 0$? [Answer: 1. $R = \emptyset$.]
   (c) $|R| = 1$? [Answer: 4. Here R must be any subset of $A \times B$ of size one. Thus $R = \{(1, 3)\}, R = \{(1, 4)\}, R = \{(2, 3)\}$, and $R = \{(2, 4)\}$.]
   (d) $|R| = 2$? [Answer: 6. $R = \{(1, 3), (1, 4)\}, R = \{(1, 3), (2, 3)\}, R = \{(1, 3), (2, 4)\}, R = \{(1, 4), (2, 3)\}, R = \{(1, 4), (2, 4)\}$, and $R = \{(2, 3), (2, 4)\}$.]
   (e) $|R| = 3$? [Answer: 4. $R = \{(1, 3), (1, 4), (2, 3)\}, R = \{(1, 3), (1, 4), (2, 4)\}, R = \{(1, 3), (2, 3), (2, 4)\}$, and $R = \{(1, 4), (2, 3), (2, 4)\}$.]
   (f) $|R| = 4$? [Answer: 1. $R = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$.]
   (g) $(1, 2) \in R$? [Answer: 0. Since $(1, 2) \notin A \times B$ there are no relations containing $(1, 2)$ here.]
   (h) $(1, 3) \in R$? [Answer: 8.]
   (i) $(1, 3) \notin R$? [Answer: 8]
7. How many relations R are there from the set $A = \{1, 2\}$ to the set $B = \emptyset$? [Answer: 1. Just the empty set.]
8. How many relations R are there from the set $A = \emptyset$ to the set $B = \{3, 4\}$? [Answer: 1. Just the empty set.]

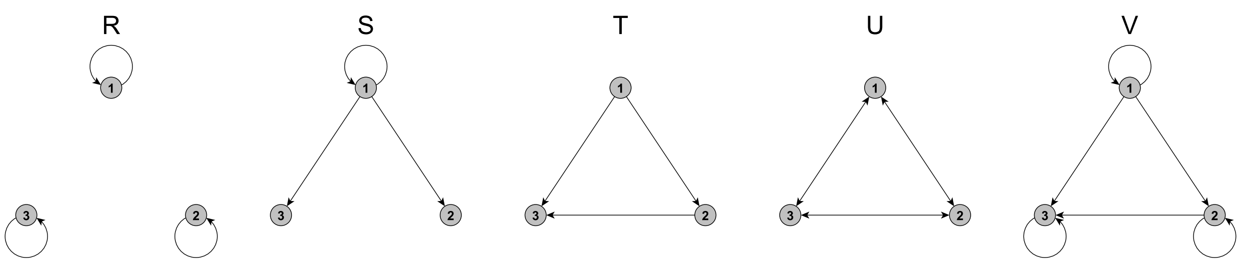
5.3 Relations and Digraphs

1. Draw a directed graph for each of the given relations on $A = \{1, 2, 3\}$.

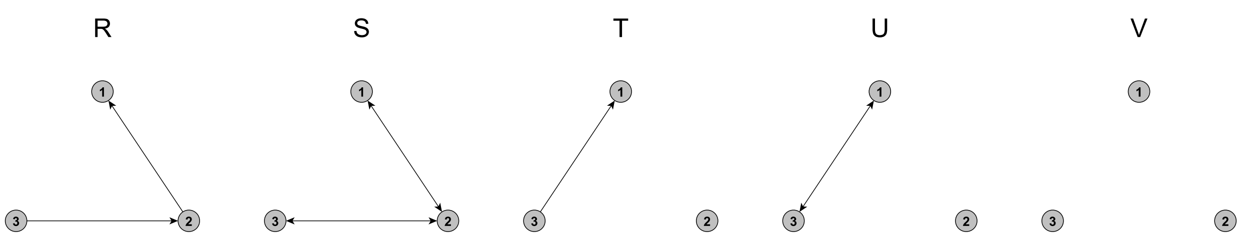
   (a) $R = A \times A, S = \emptyset, T = \{(1, 1), (2, 2)\}, U = \{(1, 1), (1, 2), (2, 1), (2, 2)\}, V = \{(1, 2), (2, 3), (3, 1)\}$.
   Answer:



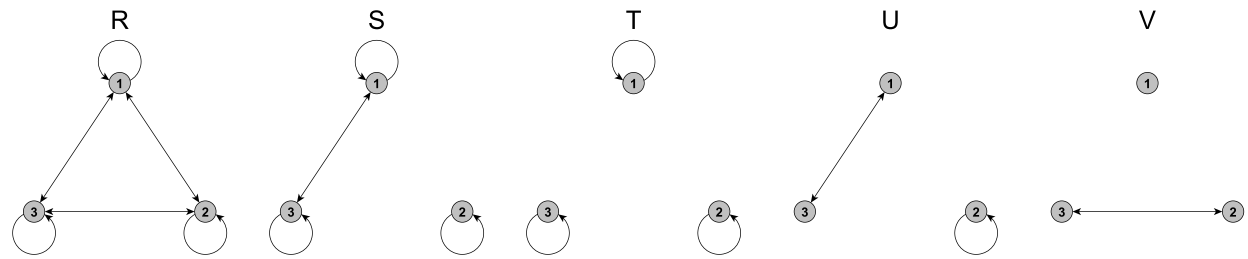
   (b) $R = \{(x, x) \in A \times A\}, S = \{(x, y) \in A \times A : x = 1\}, T = \{(x, y) \in A \times A : x < y\}, U = \{(x, y) \in A \times A : x \neq y\}, V = \{(x, y) \in A \times A : x \leq y\}$.
   Answer:



   (c) $R = \{(x, y) \in A \times A : y - x = 1\}, S = \{(x, y) \in A \times A : |y - x| = 1\}, T = \{(x, y) \in A \times A : y - x = 2\}, U = \{(x, y) \in A \times A : |y - x| = 2\}, V = \{(x, y) \in A \times A : |y - x| = 3\}$.
   Answer:

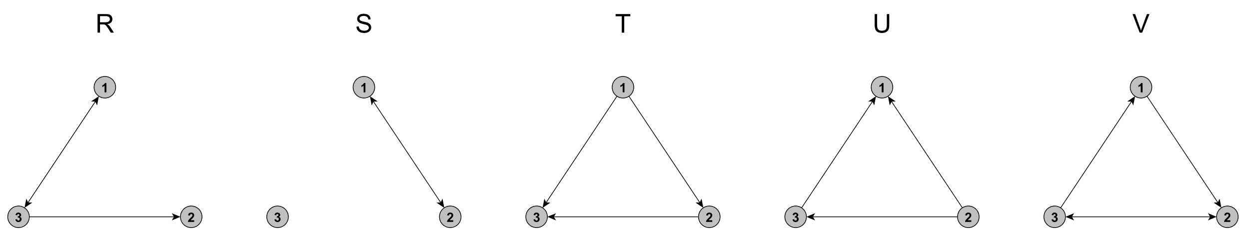


   (d) $R = \{(x, y) \in A \times A : 1 \mid y - x\}, S = \{(x, y) \in A \times A : 2 \mid y - x\}, T = \{(x, y) \in A \times A : 3 \mid y - x\}, U = \{(x, y) \in A \times A : 4 \mid x + y\}, V = \{(x, y) \in A \times A : 5 \mid x + y\}$.
   Answer:



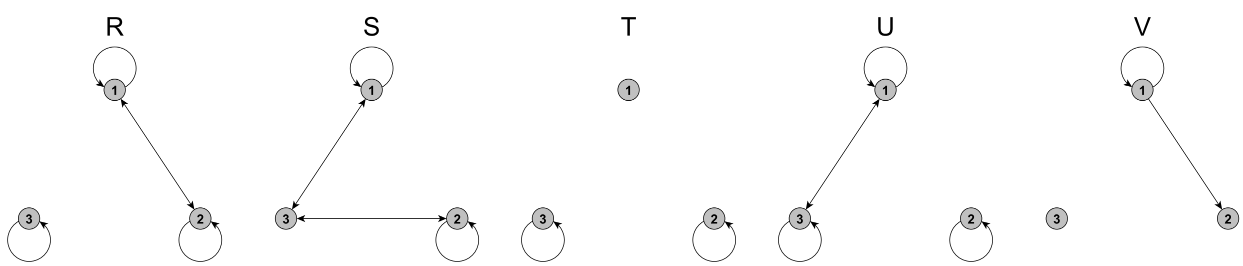
2. The following relations are given by a digraph. Write out the elements of each of these relations.

   (a) The R, S, T, U and V below.



   [Answer: $R = \{(1,3),(3,1),(3,2)\}$, $S = \{(1,2),(2,1)\}$, $T = \{(1,3),(1,2),(2,3)\}$, $U = \{(3,1),(2,1),(2,3)\}$, $V = \{(1,2),(2,3),(3,2),(3,1)\}$.

   (b) The R, S, T, U and V below.

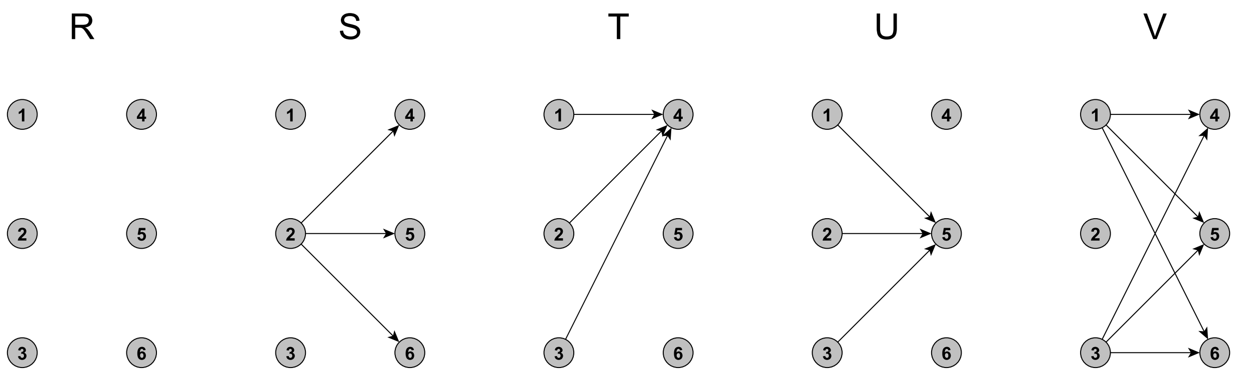


   [Answer: $R = \{(1,1),(1,2),(2,1),(2,2),(3,3)\}$, $S = \{(1,1),(1,3),(3,1),(2,2),(2,3),(3,2)\}$, $T = \{(2,2),(3,3)\}$, $U = \{(1,1),(1,3),(2,2),(3,1),(3,3)\}$, $V = \{(1,1),(1,2)\}$.

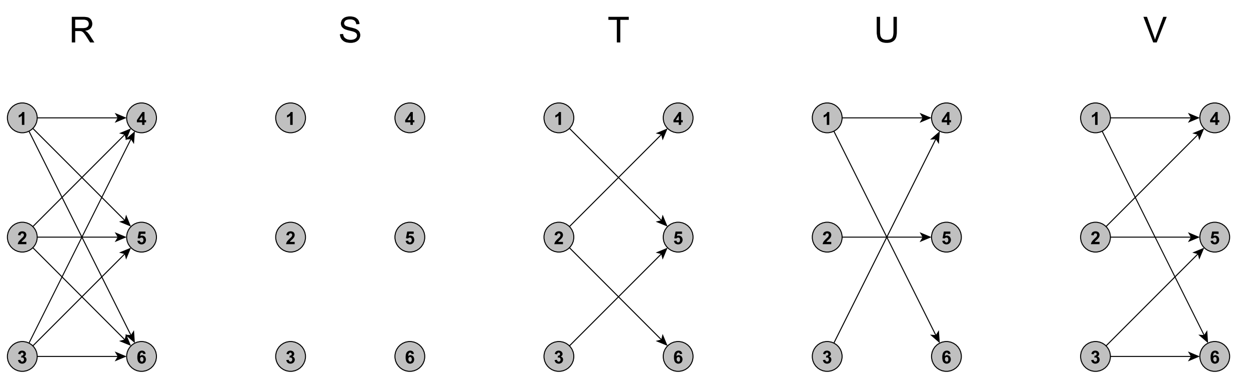
3. Draw a directed graph for each of the given relations from $A = \{1,2,3\}$ to $B = \{4,5,6\}$

   (a) $R = \{(x,x) \in A \times B\}$, $S = \{(x,y) \in A \times B : x = 2\}$, $T = \{(x,y) \in A \times B : y = 4\}$, $U = \{(x,y) \in A \times B : y \notin \mathbb{E}\}$, $V = \{(x,y) \in A \times B : x \notin \mathbb{E}\}$.

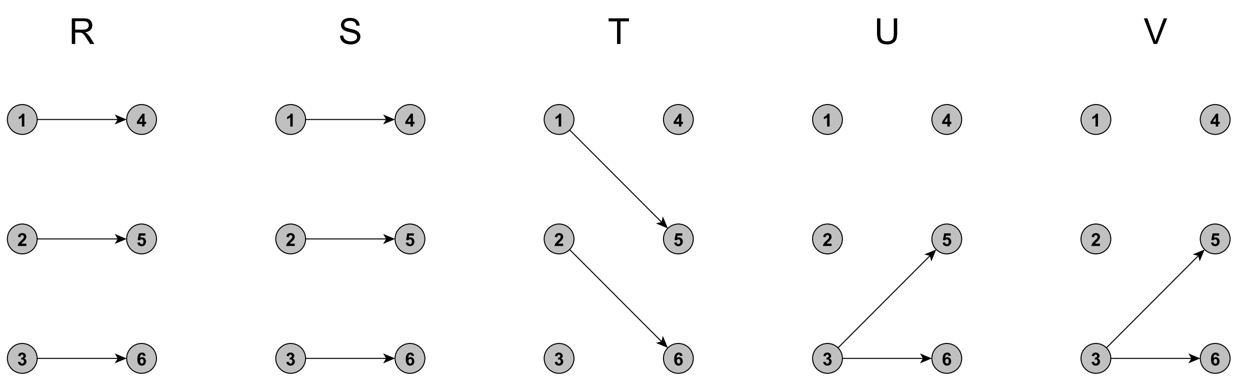
   Answer:



(b) $R = \{(x, y) \in A \times B : x < y\}, S = \{(x, y) \in A \times B : x > y\}, T = \{(x, y) \in A \times B : x + y \in \mathbb{E}\}, U = \{(x, y) \in A \times B : x + y \text{ is prime }\}, V = \{(x, y) \in A \times B : y - x \text{ is prime}\}$.
Answer:



(c) $R = \{(x, y) \in A \times B : 3 \mid (y - x)\}, S = \{(x, y) \in A \times B : 3 \mid (x - y)\}, T = \{(x, y) \in A \times B : 4 \mid (y - x)\}, U = \{(x, y) \in A \times B : x^y > 100\}, V = \{(x, y) \in A \times B : y^x > 100\}$.
Answer:


R
S
T
U
V
1
2
3
4
5
6

5.4 Inverses and Compositions of Relations

1. Find R^{-1}, dom(R^{-1}), ran(R^{-1}), $R \circ R$, dom$(R \circ R)$, and ran$(R \circ R)$, for the following relations R on the set $A = \{1, 2, 3, 4, 5\}$.

   (a) $R = \{(x, y) \in A \times A : x = y\}$. [Answer: $R^{-1} = R \circ R = R$, dom(R^{-1}) =ran(R^{-1}) =dom$(R \circ R)$ =ran$(R \circ R) = A$.]

   (b) $R = \{(x, y) \in A \times A : x = 1\}$. [Answer: $R^{-1} = \{(x, y) \in A \times A : y = 1\}, R \circ R = R$. ran$(R^{-1}) =$ dom$(R \circ R) = \{1\}$, dom$(R^{-1}) =$ ran$(R \circ R) = A$.]

   (c) $R = \{(x, y) \in A \times A : x < y\}$. [Answer: $R^{-1} = \{(x, y) \in A \times A : y < x\}$, dom$(R) = \{1, 2, 3, 4\}$, dom$(R \circ R) = \{1, 2, 3\}$, ran$(R) = \{2, 3, 4, 5\}$, ran$(R \circ R) = \{3, 4, 5\}$. $R \circ R = \{(1, 3), (2, 4), (3, 5), (1, 4), (2, 5), (1, 5)\}$.]

   (d) $R = \{(x, y) \in A \times A : x > y\}$.

   (e) $R = \{(x, y) \in A \times A : x \neq y\}$. [Answer: $R^{-1} = R, R \circ R = A \times A$, dom$(R^{-1})$ =ran(R^{-1}) =dom$(R \circ R)$ =ran$(R \circ R) = A$.]

   (f) $R = \{(x, y) \in A \times A : x \leq y\}$. [Answer: $R^{-1} = \{(x, y) \in A \times A : x \geq y\}, R \circ R = R$, dom$(R^{-1})$ =ran(R^{-1}) =dom$(R \circ R)$ =ran$(R \circ R) = A$,]

   (g) $R = \{(x, y) \in A \times A : x + y \in \mathbb{E}\}$. [Answer: $R^{-1} = R \circ R = R$, dom(R^{-1}) =ran(R^{-1}) =dom$(R \circ R)$ =ran$(R \circ R) = A$.]

   (h) $R = \{(x, y) \in A \times A : x + y \notin \mathbb{E}\}$. [Answer: $R^{-1} = R, R \circ R = \{(x, y) \in A \times A : x + y \in \mathbb{E}\}$, dom$(R^{-1})$ =ran(R^{-1}) =dom$(R \circ R)$ =ran$(R \circ R) = A$,]

   (i) $R = \{(x, y) \in A \times A : x + y$ is prime.$\}$.

   (j) $R = \{(x, y) \in A \times A : xy$ is prime.$\}$. [Answer: $R^{-1} = R$. $R \circ R = \{(1, 1), (2, 2), (3, 3), (5, 5), (5, 2), (2, 5)\ (3, 2), (2, 3), (3, 5), (5, 3)\}$, dom$(R^{-1})$ =ran(R^{-1}) =dom$(R \circ R)$ =ran$(R \circ R) = \{1, 2, 3, 5\}$.]

   (k) $R = \{(x, y) \in A \times A : y - x$ is prime.$\}$.

   (l) $R = \{(x, y) \in A \times A : x \mid y\}$.

   (m) $R = \{(x, y) \in A \times A : y - x = 1\}$.

   (n) $R = \{(x, y) \in A \times A : |y - x| = 1\}$.

   (o) $R = \{(x, y) \in A \times A : y - x = 2\}$. [Answer: $R^{-1} = \{(x, y) \in A \times A : x - y = 2\}, R \circ R = \{(1, 5)\}$, dom$(R^{-1}) = \{3, 4, 5\}$, ran$(R^{-1}) = \{1, 2, 3\}$, dom$(R \circ R) = \{1\}$, ran$(R \circ R) = \{5\}$.]

   (p) $R = \{(x, y) \in A \times A : x - y = 2\}$.

   (q) $R = \{(x, y) \in A \times A : |y - x| = 3\}$. [Answer: $R^{-1} = R, R \circ R = \{(1, 1), (2, 2), (4, 4), (5, 5)\}$, ran$(R^{-1}) = \{1, 2, 4, 5\} =$ dom$(R^{-1}) =$ dom$(R \circ R) =$ ran$(R \circ R)$.]

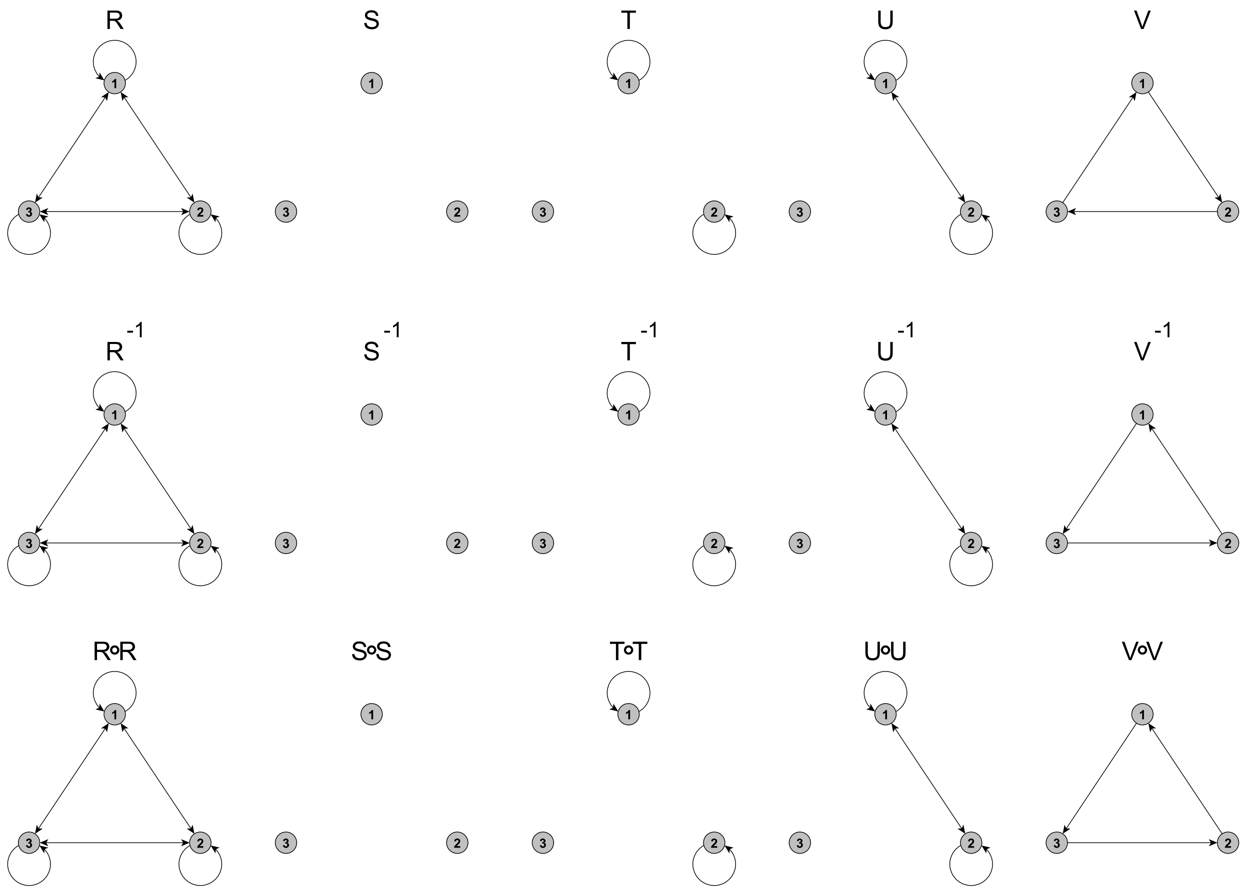
   (r) $R = \{(x, y) \in A \times A : |y - x| = 4\}$.

   (s) $R = \{(x, y) \in A \times A : y = x^2\}$.

   (t) $R = \{(x, y) \in A \times A : x = y^2\}$.

   (u) $R = \{(x, y) \in A \times A : xy = 4\}$. [Answer: $R^{-1} = R, R \circ R = \{(1, 1), (2, 2), (4, 4)\}$, ran$(R^{-1}) = \{1, 2, 4\} =$ dom$(R^{-1}) =$ dom$(R \circ R) =$ ran$(R \circ R)$.]

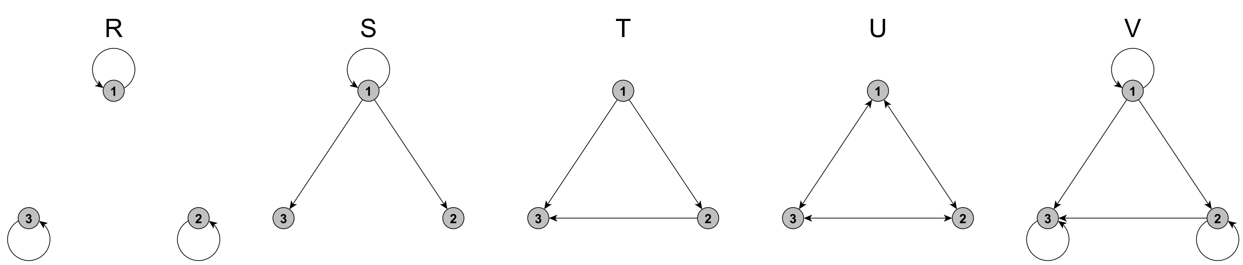
2. For each of the following relations find the inverse and composition of each with itself.



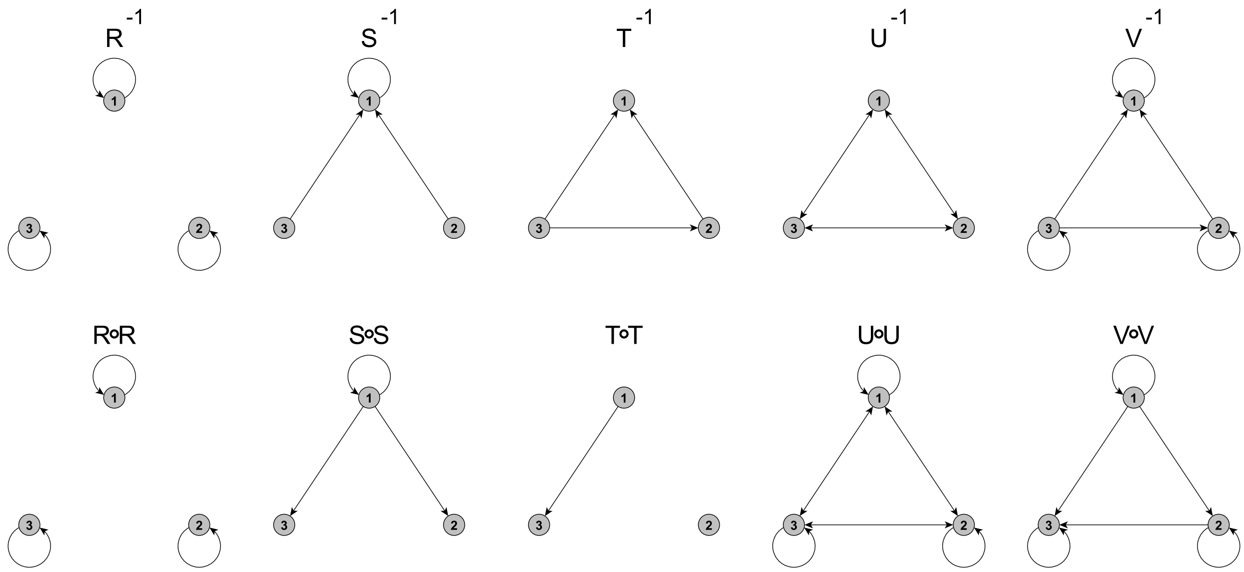
(a) R, S, T, U and V are as follows:

Answer:

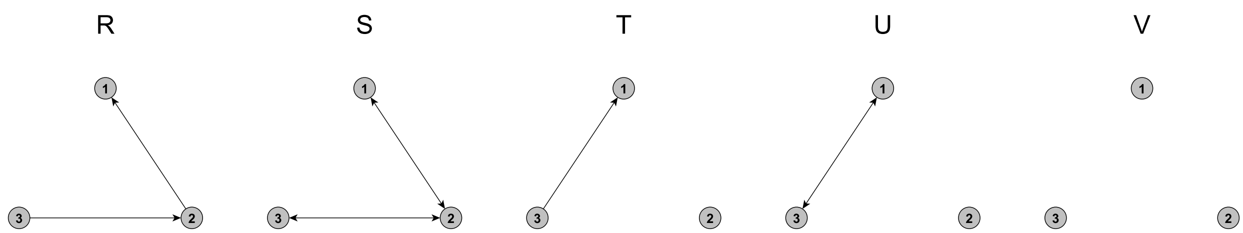
(b) R, S, T, U and V are as follows:



Answer:



(c) R, S, T, U and V are as follows:



Answer:

3. Let R be a relation from the set $B = \{4, 5, 6\}$ to $C = \{7, 8, 9\}$ and S be a relation from the set $A = \{1, 2, 3\}$ to the set B. Find $R \circ S$, dom $R \circ S$ and ran $R \circ S$ in the following situations.

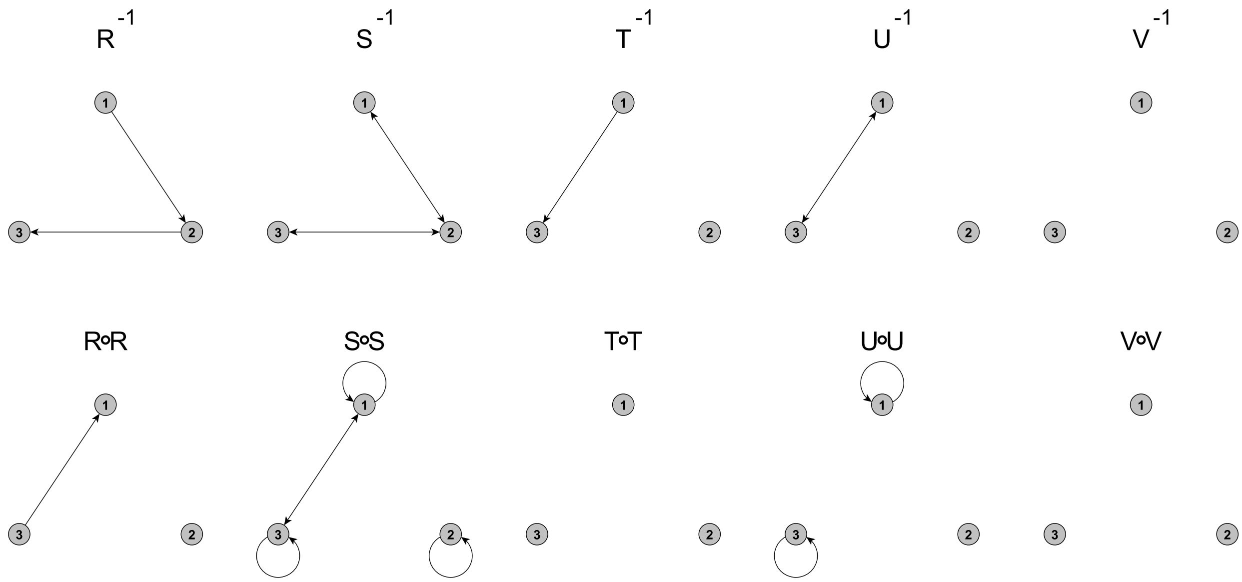
   (a) $R = \{(4, 7), (5, 8), (6, 9)\}, S = \{(1, 4), (2, 5), (3, 6)\}$. [Answer: $R \circ S = \{(1, 7), (2, 8), (3, 9)\}$, $\mathrm{dom}(R \circ S) = A$, $\mathrm{ran}(R \circ S) = C$.]

   (b) $R = \{(4, 7), (5, 8)\}, S = \{(2, 5), (3, 6)\}$. [Answer: $R \circ S = \{(2, 8)\}$, $\mathrm{dom}(R \circ S) = \{2\}$, $\mathrm{ran}(R \circ S) = \{8\}$.]

   (c) $R = \{(4, 7), (5, 8)\}, S = \{(2, 5), (3, 4)\}$. [Answer: $R \circ S = \{(2, 8), (3, 7)\}$, $\mathrm{dom}(R \circ S) = \{2, 3\}$, $\mathrm{ran}(R \circ S) = \{7, 8\}$.]

   (d) $R = \{(4, 7), (5, 7), (6, 7)\}, S = \{(1, 4), (2, 5), (3, 4)\}$. [Answer: $R \circ S = \{(1, 7), (2, 7), (3, 7)\}$, $\mathrm{dom}(R \circ S) = A$, $\mathrm{ran}(R \circ S) = \{7\}$.]

   (e) $R = \{(5, 7), (5, 8), (5, 9)\}, S = \{(1, 4), (2, 5), (3, 4)\}$. [Answer: $R \circ S = \{(2, 7), (2, 8), (2, 9)\}$, $\mathrm{dom}(R \circ S) = \{2\}$, $\mathrm{ran}(R \circ S) = C$.]



(f) $R = \{(5,7),(5,8),(5,9),(6,7),(6,8),(6,9)\}, S = \{(1,4),(2,4),(3,4)\}$. [Answer: $R \circ S = \emptyset$, $\operatorname{dom}(R \circ S) = \emptyset$, $\operatorname{ran}(R \circ S) = \emptyset$.]

(g) $R = \{(4,7),(4,8),(4,9),(6,7),(6,8),(6,9)\}, S = \{(1,4),(2,4),(3,4),(1,6),(2,6),(3,6)\}$. [Answer: $R \circ S = A \times C$, $\operatorname{dom}(R \circ S) = A$, $\operatorname{ran}(R \circ S) = C$.]

(h) $R = \{(4,7),(4,8),(4,9)\}, S = \{(1,4),(2,4),(3,4)\}$. [Answer: $R \circ S = A \times C$, $\operatorname{dom}(R \circ S) = A$, $\operatorname{ran}(R \circ S) = C$.]

(i) $R = \{(4,7),\ (4,8),\ (4,9),\ (5,7),\ (5,8),\ (6,7),\ (6,8),\ (6,9)\}, S = \{(1,4),\ (1,5),\ (1,6),\ (2,4),\ (2,5),\ (2,6),\ (3,5)\}$. [Answer: $R \circ S = \{(1,7),\ (1,8),\ (1,9),\ (2,7),\ (2,8),\ (2,9),\ (3,7),\ (3,8)\}$, $\operatorname{dom}(R \circ S) = A$, $\operatorname{ran}(R \circ S) = C$.]

(j) $R = \{(4,8),\ (5,9)\}, S = \{(1,5),\ (2,6)\}$. [Answer: $R \circ S = \{(1,9)\}$, $\operatorname{dom}(R \circ S) = \{1\}$, $\operatorname{ran}(R \circ S) = \{9\}$.]

(k) $R = \{(4,8),\ (5,9),\ (6,7)\}, S = \{(1,5),\ (2,6),\ (3,4)\}$. [Answer: $R \circ S = \{(1,9),(2,7),(3,8)\}$, $\operatorname{dom}(R \circ S) = A$, $\operatorname{ran}(R \circ S) = C$.]

5.5 Properties of Relations and Constructions

1. Find a relation on the set $\{1, 2\}$ which has the following properties. There will generally be multiple answers.

   (a) R is reflexive, symmetric, and transitive. [Answer: $R = \{(1, 1), (2, 2)\}$.]

   (b) R is reflexive and transitive, but not symmetric. [Answer: $R = \{(1, 1), (1, 2), (2, 2)\}$.]

   (c) R is not reflexive, but is symmetric and transitive. [Answer: $R = \emptyset$.]

   (d) R is not reflexive or transitive, but is symmetric. [Answer: $R = \{(1, 2), (2, 1)\}$.]

   (e) R is not reflexive or symmetric, but is transitive. [Answer: $R = \{(1, 2)\}$.]

2. Find a relation on the set $\{1, 2, 3\}$ which has the following properties. There will generally be multiple answers.

   (a) R is reflexive, symmetric, and transitive. [Answer: $R = \{(1, 1), (2, 2), (3, 3)\}$.]

   (b) R is reflexive, symmetric, and not transitive. [Answer: $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$.]

   (c) R is reflexive and transitive but, not symmetric. [Answer: $R = \{(1, 1), (1, 2), (2, 2), (3, 3)\}$.]

   (d) R is reflexive, but not symmetric and not transitive. [Answer: $R = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\}$.]

   (e) R is not reflexive, but is both symmetric and transitive. [Answer: $R = \emptyset$.]

   (f) R is not reflexive, R is symmetric, and R is not transitive. [Answer: $R = \{(1, 2), (2, 1)\}$.]

   (g) R is not reflexive or symmetric, but R is transitive. [Answer: $R = \{(1, 2)\}$.]

   (h) R is not reflexive, not symmetric, and not transitive. [Answer: $R = \{(1, 2), (2, 3)\}$.]

3. Let R be the smallest equivalence relation on A containing the listed elements[1]. State whether or not R is $A \times A$.

   (a) $A = \{1, 2\}$, $(1, 2) \in R$ [Answer: $R \neq A \times A$.]

   (b) $A = \{1, 2, 3\}$, $(1, 2) \in R$ [Answer: $R \neq A \times A$.]

   (c) $A = \{1, 2, 3\}$, $\{(1, 2), (2, 3)\} \subseteq R$. [Answer: $R = A \times A$.]

   (d) $A = \{1, 2, 3\}$, $\{(1, 2), (2, 1)\} \subseteq R$. [Answer: $R \neq A \times A$.]

   (e) $A = \{1, 2, 3, 4\}$, $\{(1, 2), (2, 3)\} \subseteq R$. [Answer: $R \neq A \times A$.]

   (f) $A = \{1, 2, 3, 4\}$, $\{(1, 2), (3, 4)\} \subseteq R$. [Answer: $R \neq A \times A$.]

   (g) $A = \{1, 2, 3, 4\}$, $\{(1, 2), (1, 4)\} \subseteq R$. [Answer: $R \neq A \times A$.]

   (h) $A = \{1, 2, 3, 4\}$, $\{(1, 2), (2, 3), (3, 4)\} \subseteq R$ [Answer: $R = A \times A$.]

4. How many relations R are there on the set $A = \{1, 2\}$ with the following properties?

   (a) R is reflexive [Answer: 4. $\{(1, 1), (2, 2)\}$, $\{(1, 1), (2, 2), (2, 1)\}$, $\{(1, 1), (2, 2), (2, 1)\}$, $\{(1, 1), (2, 2), (1, 2), (2, 1)\}$]

   (b) R is not reflexive [Answer: 12. There are sixteen relations on a set with two elements and $16 - 4 = 12$.]

[1] By this we mean the relation with the fewest elements of $A \times A$.

(c) R is symmetric [Answer: 8. We have to include both $(1,2)$ and $(2,1)$ or neither. We can just count the subsets of $\{(1,1),(1,2),(2,2)\}$ since we no longer get a choice for $(2,1)$ after $(1,2)$ has been decided. There are eight subsets of any three element set.]

(d) R is not symmetric [Answer: 8. 16-8=8.]

(e) R is reflexive and not symmetric. [Answer: 2. We have to include either $(1,2)$ or $(1,2)$ but not both or it will be symmetric. We also must include both $(1,1)$ and $(2,2)$. This leaves $\{(1,1),(1,2),(2,2)\}$ and $\{(1,1),(2,1),(2,2)\}$ for our only choices.

(f) R is symmetric and transitive. [Answer: 5. If $(1,2)$ is included we would need $(2,1)$ to get symmetry, and transitivity forces us to include all of $A \times A$. This is one possibility. If $(1,2)$ is not included then we can't include $(2,1)$ leaving us with the four possibilities $\emptyset$, $\{(2,2)\}$, $\{(1,1)\}$, and $\{(1,1),(2,2)\}$.

(g) R is symmetric and not transitive. [Answer: 2. We have to include either $(1,2)$ or $(1,2)$ otherwise it will be transitive. By symmetry, we will have to include both. Then we have to choose whether to include $(1,1)$ and $(2,2)$. If we include both or neither it is transitive, therefore we must include exactly one. Our possibilities are $\{(1,2),(2,1),(2,2)\}$ and $\{(1,1),(1,2),(2,1)\}$.]

(h) R is not symmetric and not transitive. [Answer: 0. We have to include either $(1,2)$ or $(1,2)$ but not both or it will be symmetric. Regardless of whether we include $(1,1)$ or $(2,2)$ it is now transitive so no relations meet these properties on our set.]

(i) R is reflexive and not transitive. [Answer: 0. We must include $(1,1)$ and $(1,2)$. If we add either $(1,2)$ or $(2,1)$ it will be transitive. If we add both or neither it will still be transitive.]

5. How many distinct equivalence relations R on $A = \{1,2,3\}$ have the following sizes?

   (a) $|R| = 3$ [Answer: 1.]

   (b) $|R| = 4$ [Answer: 0.]

   (c) $|R| = 5$ [Answer: 3.]

   (d) $|R| = 6$ [Answer: 0.]

   (e) $|R| = 7$ [Answer: 0.]

   (f) $|R| = 9$ [Answer: 1.]

5.6 Proofs with Properties of Relations

For each of the following relations state whether they are reflexive, symmetric, and transitive[2]. Support each answer with either a proof or a counterexample.

1. The relation R on $\mathbb{N}$ given by $x \sim y$ iff $x = 2y$.
2. The relation R on $\mathbb{N}$ given by $x \sim y$ iff $2x \leq y$.

   Proof:

   Reflexivity: It is not reflexive because when $a = 1$, $a \nleq a$ so $a \nsim a$.

   Symmetric: If $a = 1$ and $b = 2$ then $a \sim b$ because $2 \leq 2$, yet $b \nsim a$ since four is not less than or equal to one.

   Transitive: If $a \sim b$ and $b \sim c$ then $2a \leq b$ and $2b \leq c$. Thus $4a \leq 2b \leq c$. Now since a is positive[3], $2a < 4a$. Thus $2a \leq c$ which shows $a \sim c$. $\square$
3. The relation R on $\mathbb{N}$ given by $x \sim y$ iff x^y is even.

   The relation is transitive but not reflexive or symmetric.

   Proof:

   Reflexive: It is not reflexive as $(1, 1) \notin R$ since $1^1 = 1$ is not even.

   Symmetric: It is not symmetric as $(2, 3) \in R$ but $(3, 2) \notin R$.

   Transitive: It is transitive. If $(x, y) \in R$ and $(y, z) \in R$ then x^y is even which means x is even. Thus x^z is even so $(x, z) \in R$. $\square$
4. The relation R on $\mathbb{N}$ given by $x \sim y$ iff $x + y$ is prime.
5. The relation R on $\mathbb{N}$ given by $x \sim y$ iff xy is prime.
6. The relation R on $\mathbb{Z}$ given by $x \sim y$ iff $x + y = 0$.
7. The relation R on $\mathbb{Z}$ given by $x \sim y$ iff $xy = 0$.
8. The relation R on $\mathbb{Z}$ given by $x \sim y$ iff y is the successor of x. (That means $y = x + 1$.)

   The relation is reflexive, but not symmetric and not transitive.

   Proof:

   Reflexivity: Since $0 < 1$ we know $a < a + 1$ for any a. This means $a \sim a$.

   Symmetry: When $a = 1$ and $b = 2$ we have $a \sim b$ because $b = a + 1$, but not $b \sim a$.

   Transitivity: When $a = 1$, $b = 2$ and $c = 3$ we have $a \sim b$ and $b \sim c$ but $a \nsim c$ because $3 \neq 1 + 1$. $\square$

[2] Keep in mind that these questions are using the " $\sim$" notation for relations, so that $a \sim b$ if and only if $(a, b) \in R$. Thus R is reflexive if $a \sim a$ for each a in our underlying set, which is equivalent to saying $(a, a) \in R$ for each a in that set. We know R is symmetric if $(a, b) \in R \Rightarrow (b, a) \in R$, which is equivalent to saying that $a \sim b$ implies $b \sim a$. Finally R is transitive if $(a, b) \in R \wedge (b, c) \in R$ imply $(a, c) \in R$, or that $a \sim b$ and $b \sim c$ imply $a \sim c$. Though the questions are phrased using one notation, feel free to use whichever you prefer in order to solve these problems.

[3] Notice that this statement, fails to be true if we extend our set to the integers. It is important to pay close attention to which set we are working over in each example.

9. The relation R on $\mathbb{Z}$ given by $x \sim y$ iff one is a successor of the other. (That means $x = y+1$ or $y = x+1$.)

   This relation is symmetric but not reflexive or transitive.

   Proof:

   Reflexivity: It is not reflexive as 1 is not equal to $1+1$.

   Symmetry: It is symmetric as $(x, y) \in R$ implies either $x = y+1$ or $y = x+1$ which implies $y = x+1$ or $x = y+1$ so $(y, x) \in R$.

   Transitivity: It is not transitive as $x = 1, y = 2, z = 3$ is a counterexample. $(1, 2) \in R$ and $(2, 3) \in R$ but $(1, 3) \notin R$. $\square$

10. The relation R on $\mathbb{Z}$ given by $x \sim y$ iff $xy \geq 0$.

    The relation is reflexive and symmetric but not transitive.

    Proof:

    Reflexive: It is reflexive as for any x, $x \cdot x = x^2 \geq 0$.

    Symmetric: It is symmetric as $xy \geq 0$ implies $yx \geq 0$ since $xy = yx$.

    Transitive: It is not transitive as $x = -1, y = 0, z = 1$ is a counterexample. This is because $-1 \cdot 0 \geq 0$ and $0 \cdot 1 \geq 0$. $\square$

11. The relation R on $\mathbb{Z}$ given by $x \sim y$ iff $x + y$ is even.

12. The relation R on $\mathbb{Z}$ given by $x \sim y$ iff xy is even.

13. The relation R on $\mathbb{Z}$ given by $x \sim y$ iff xy is odd.

    The relation is not reflexive, but is both symmetric and transitive.

    Proof:

    Reflexive: When $a = 0$ we do not have $a \sim a$ because $a^2 = 0$ which is not odd.

    Symmetric: If $a \sim b$ then ab is odd. As $ab = ba$ we know ba is odd, and therefore $b \sim a$.

    Transitive: Suppose $a \sim b$ and $b \sim c$. Then we know ab and bc are both odd. We know that ab being odd implies that a and b are odd[4] Similarly we get that b and c are both odd. Thus a, b, and c are all odd. proving that ac is odd, and thus $a \sim c$. $\square$

14. The relation R on $\mathbb{Z}$ given by $x \sim y$ iff $x + y$ is divisible by 3.

    The relation is symmetric, but neither reflexive nor transitive.

    Proof:

    Reflexivity: When $a = 1$ we fail to get $1 \sim 1$ because two is not divisible by three.

    Symmetry: If $a \sim b$ we know three divides $a + b$ which equals $b + a$. Thus three divides $b + a$ and we know $b \sim a$.

    Transitive: If $a = 1, b = 2$ and $c = 1$ we know $a \sim b$ and $b \sim c$ because three divides three. However, $a \nsim c$ because three does not divide $1 + 1$. $\square$

[4] This comes from an older proof we did. Even if we've forgotten this fact, taking the contrapositive allows us to do this with a very short proof. We also proved that a product of two odd numbers was odd, which is a short direct proof that we are about to use at the end.

15. The relation R on $\mathbb{Z}$ given by $x \sim y$ iff $x - y$ is divisible by 3.

16. The relation R on $\mathbb{Z}$ given by $x \sim y$ iff the average of x and y is in $\mathbb{Z}$. (Note that the average equals $\frac{1}{2}(x+y)$.)

    The relation is reflexive, symmetric and transitive and hence is an equivalence relation.

    Proof:

    Reflexivity: It is reflexive as for any x, $\frac{1}{2}(x+x) = \frac{1}{2}2x = x \in \mathbb{Z}$.

    Symmetry: It is symmetric as $\frac{1}{2}(x+y) = \frac{1}{2}(y+x)$ so $(x,y) \in R$ implies $(y,x) \in R$.

    Transitivity: It is transitive. Assume (x,y) and $(y,z) \in R$. Then $\frac{1}{2}(x+y) = m$ and $\frac{1}{2}(y+z) = n$ for some $m,n \in \mathbb{Z}$. Then $\frac{1}{2}(x+z) = \frac{1}{2}(x+y+y+z-2y) = \frac{1}{2}(x+y) + \frac{1}{2}(y+z) - \frac{1}{2}2y = m+n-y$ which is in $\mathbb{Z}$. $\square$

17. The relation R on $\mathbb{Z}$ given by $x \sim y$ iff $x \leq 2y$.

    The relation is not reflexive, symmetric or transitive.

    Proof:

    Reflexivity: When $a = -1$ $a \nsim a$ because $-1 \nleq -2$.

    Symmetry: When $a = 1$ and $b = 3$ we have $a \sim b$ but not $b \sim a$ because $3 \nleq 2 \times 1$.

    Transitivity: When $a = 4, b = 2$ and $c = 1$ we know $a \sim b$ and $b \sim c$ because $a \leq 2b$ and $b \leq 2c$. Yet $a \nsim c$ because $4 \nleq 2 \times 1$. Thus the relation is not transitive. $\square$

18. The relation R on $\mathbb{Z}$ given by $x \sim y$ iff $x < y - 2$.

    The relation is not reflexive or symmetric, but is transitive.

    Proof:

    Reflexivity: Zero is not less than 0-2, thus $(0,0) \notin R$.

    Symmetry: $0 \sim 100$ but it is not true that $100 \sim 0$ since 100 is not less than 0-2.

    Transitivity: Assume that $a \sim b$ and $b \sim c$. Thus $a < b - 2$ and $b < c - 2$ so we know for certain that $a < (c-2) - 2 = c - 4$. Since $a < c - 4$ we know $a < c - 2$ and thus $a \sim c$. $\square$

19. The relation R on $\mathbb{E}$ given by $x \sim y$ iff six divides $y - x$.

20. The relation R on $\mathbb{E}$ given by $x \sim y$ iff four divides $x + y$.

21. The relation R on $\mathbb{E}$ given by $x \sim y$ iff six divides $x + y$.

    The relation is not reflexive or transitive, but is symmetric.

    Proof:

    Reflexivity: Six does not divide two plus two so $(2,2) \notin R$.

    Symmetry: Suppose $(a,b) \in R$. Then six divides $a+b$ so $6r = a+b$ for $r \in \mathbb{Z}$. Then $6r$ also equals $b+a$, which shows that six divides $b+a$ and thus $(b,a) \in R$.

    Transitivity: $(4,2)$ and $(2,10)$ are in R because six divides $4+2$ and $2+10$. However $(4,10) \notin R$ because six does not divide fourteen. $\square$

22. The relation R on $\mathbb{E}$ given by $x \sim y$ iff four divides x.

23. The relation R on $\mathbb{Q}$ given by $x \sim y$ iff $xy \in \mathbb{Z}$.

24. The relation R on $\mathbb{Q}$ given by $x \sim y$ iff $x + y \in \mathbb{Z}$.

25. The relation R on $\mathbb{Q}$ given by $x \sim y$ iff $x - 2y \in \mathbb{Z}$.

    The relation is not reflexive, not symmetric and not transitive.

    Proof:

    Reflexivity: If $x = \frac{1}{2}$ then $(x, x) \notin R$ as $\frac{1}{2} - 2 \cdot \frac{1}{2} = -\frac{1}{2}$ which is not an integer.

    Symmetry: If $x = \frac{3}{2}$ and $y = \frac{1}{4}$ then $x \sim y$ because $\frac{3}{2} - 2 \cdot \frac{1}{4} = 1$ which is in $\mathbb{Z}$. We do not have $y \sim x$ however, because $\frac{1}{4} - 2 \cdot \frac{5}{4} = -\frac{9}{4}$, which is not an integer.

    Transitivity: If $x = \frac{1}{2}, y = \frac{1}{4}$ and $z = \frac{1}{8}$ then $x \sim y$ and $y \sim z$. However it is not true that $x \sim z$ because $\frac{1}{2} - 2 \cdot \frac{1}{8} = \frac{1}{4}$ which is not an integer. $\square$

26. The relation R on $\mathbb{Q}$ given by $x \sim y$ iff $xy = 2$.

    The relation is symmetric but not reflexive or transitive.

    Proof:

    Reflexivity: It is not reflexive. For example $(\frac{1}{2}, \frac{1}{2}) \notin R$ as $\frac{1}{2} \cdot \frac{1}{2} \neq 2$

    Symmetry: It is symmetric. If $(x, y) \in R$ then $xy = 2$ so $yx = 2$ and $(y, x) \in R$.

    Transitivity: It is not transitive. For example $x = \frac{2}{3}, y = \frac{6}{2}, z = \frac{2}{3}$ is an example where xRy and yRz but $(x, z) \notin R$ as as $\frac{2}{3} \cdot \frac{2}{3} \neq 2$ $\square$

27. The relation R on $\mathbb{R}$ given by $u \sim v$ iff $u^2 = v^2$.

    The relation is reflexive, symmetric and transitive, and hence it is an equivalence relation.

    Proof:

    Reflexivity: For any real number x, we know $x^2 = x^2$ and thus $x \sim x$.

    Symmetry: If $x \sim y$ then $x^2 = y^2$, thus $y^2 = x^2$ and so $y \sim x$.

    Transitivity: If $x \sim y$ and $y \sim z$ then $x^2 = y^2$ and $y^2 = z^2$. Therefore $x^2 = y^2 = z^2$ and $x \sim z$. $\square$

28. The relation R on $\mathbb{R}$ given by $u \sim v$ iff $v - u < 1$.

    The relation is reflexive, but not symmetric or transitive.

    Proof:

    Note that we can rewrite the condition as $x - y < 1$ iff $x < 1 + y$.

    Reflexivity: For any real number x, we know $x < 1 + x$ because $0 < 1$. Thus $x \sim x$.

    Symmetry: If $x = 1$ and $y = 100$ then $x < y + 1$ but it is not true that $y < x + 1$. Thus the relation is not symmetric.

    Transitivity: If $x < y + 1$ and $y < z + 1$ we know $x < z + 1 + 1$ which means $x < z + 2$. This doesn't mean that $x < z + 1$ though. Consider the case where $x = 1, y = \frac{1}{2}$, and $z = 0$. Then $x \sim y$ and $y \sim z$ because $1 < \frac{1}{2} + 1$ and $\frac{1}{2} < 0 + 1$, but one is not less than 0+1 since the two are equal. This shows the relation is not transitive. $\square$

29. The relation R on $\mathbb{R}$ given by $u \sim v$ iff $u - v > -1$.

    The relation is reflexive but neither symmetric nor transitive.

    Proof:

    Reflexivity: For any real number x we know $x - x = 0 > -1$, and thus $x \sim x$.

    Symmetry: If $x = 10$ and $y = 1$ then $u \sim v$ because $10 - 1 > -1$, but $v \nsim u$ because $1 - 10 = -9$ which is not bigger than -1.

    Transitivity: If $x = -\frac{1}{2}, y = 0$ and $z = \frac{1}{2}$ then $x - y$ and $y - z$ equal $-\frac{1}{2}$. Thus $x \sim y$ and $y \sim z$. However $x - z = -\frac{1}{2} - \frac{1}{2} = 1$, which is not stricktly less than negative one. Therefore $x \nsim z$. $\square$

30. The relation R on $\mathbb{R}$ given by $u \sim v$ iff $\sqrt{u^2 + v^2} < 1$.

31. The relation R on $\mathbb{R}$ given by $u \sim v$ iff $u^2 + v^2 > 0$.

32. The relation[5] R on $\mathbb{Z} \times \mathbb{Z}$ given by $(a, b) \sim (c, d)$ iff $b = d$.

    The relation is reflexive, symmetric and transitive and hence is an equivalence relation.

    Proof:

    Reflexivity: The relation is reflexive. $(w, x) \sim (w, x)$ for any $(w, x) \in \mathbb{Z} \times \mathbb{Z}$ because x is always equal to x.

    Symmetry: The relation is symmetric. If $(w, x) \sim (y, z)$ then $x = z$ so $z = x$ which means $(y, z) \sim (w, x)$.

    Transitivity: The relation is transitive. Suppose $(u, v) \sim (w, x)$ and $(w, x) \sim (y, z)$ for the ordered pairs $(u, v), (w, x)$ and (y, z). Then we know $v = x$ and $x = z$. This implies $v = z$ so $(u, v) = (y, z)$. $\square$

33. The relation R on $\mathbb{Z} \times \mathbb{Z}$ given by $(a, b) \sim (c, d)$ iff $a + c = 0$ and $b + d = 0$.

34. The relation R on $\mathbb{Z} \times \mathbb{Z}$ given by $(a, b) \sim (c, d)$ iff $a + b = 0$ and $c + d = 0$.

35. The relation R on $\mathbb{Z} \times \mathbb{Z}$ given by $(a, b) \sim (c, d)$ iff $a + b + c + d = 0$.

36. The relation R on $\mathbb{Z} \times \mathbb{Z}$ given by $(a, b) \sim (c, d)$ iff $a^2 = c^2$ and $b^2 = d^2$.

    The relation is Proof:

    Reflexivity: $(x, y) \sim (x, y)$ for all $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ because $x^2 = x^2$ and $y^2 = y^2$.

    Symmetry: Suppose $(w, x) \sim (y, z)$ so $w^2 = y^2$ and $x^2 = z^2$. As $y^2 = w^2$ and $z^2 = x^2$ we know $(y, z) \sim (w, x)$.

    Transitivity: Suppose $(u, v) \sim (w, x)$ and $(w, x) \sim (y, z)$. Then $u^2 = w^2$ and $v^2 = x^2$ and $w^2 = y^2$ and $x^2 = z^2$. Putting this all together tells us $u^2 = y^2$ and $v^2 = z^2$, which means $(u, v) \sim (y, z)$. $\square$

37. The relation R on $\mathbb{Z} \times \mathbb{Z}$ given by $(a, b) \sim (c, d)$ iff $abcd = 0$.

[5] Keep in mind that we are now considering a relation on the set $\mathbb{Z} \times \mathbb{Z}$, which means this is a subset of $(\mathbb{Z} \times \mathbb{Z}) \times (\mathbb{Z} \times \mathbb{Z})$. Here $(a, b) \sim (c, d)$ if and only iff $((a, b), (c, d)) \in R$.

38. The relation R on the alphabet $\{a, b, \cdots, z\}$ given by $\alpha \sim \beta$ iff there is a vowel between α and β.

    The relation is symmetric but not reflexive or transitive.

    Proof:

    Reflexivity: It is not reflexive as $(b, b) \notin R$.

    Symmetry: It is symmetric. Suppose $(\alpha, \beta) \in R$. Then there is a vowel between α and β so the same vowel is in between β and α. This means $(\beta, \alpha) \in R$.

    Transitivity: It is not transitive because (b, f) and (f, c) are in R, but (b, c) is not. $\square$

39. The relation R on the alphabet $\{a, b, \cdots, z\}$ given by $\alpha \sim \beta$ iff α is the letter before β.

40. The relation R on the alphabet $\{a, b, \cdots, z\}$ given by $\alpha \sim \beta$ iff α is in the same half of the alphabet as β. (Here note that $\{a, b, \cdots, m\}$ and $\{n, o, \cdots, z\}$ are the two halves in question.)

41. The relation R on the alphabet $\{a, b, \cdots, z\}$ given by $\alpha \sim \beta$ iff α and β are both consonants.

    The relation is symmetric and transitive but not reflexive.

    Proof:

    Reflexivity: It is not reflexive as $(a, a) \notin R$.

    Symmetry: It is symmetric. If $(\alpha, \beta) \in R$ then both α and β are consonants. Thus β and α are both consonants and $(\beta, \alpha) \in R$.

    Transitivity: It is transitive. If $(\alpha, \beta) \in R$ and $(\beta, \gamma) \in R$ then all three must be consonants. Thus $(\alpha, \gamma) \in R$. $\square$

5.7 Partitions

1. Which of the following are partitions of the set $\{1, 2, 3, 4, 5, 6\}$?

   (a) $\{1, 2, 3, 4, 5, 6\}$
   (b) $\{\{1, 2, 3, 4, 5, 6\}\}$
   (c) $\{\{1\}, \{2\}, \{3\}\}$
   (d) $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$
   (e) $\{\{1, 2, 3\}, \{3, 4, 5\}, \{4, 5, 6\}\}$
   (f) $\{\{1, 2, 3\}, \{4, 5, 6\}\}$
   (g) $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$
   (h) $\{\{1, 2\}, \{3\}, \{4\}, \{5, 6\}\}$

   Answer: all but a, c, e.

2. Which of the following are partitions of the set $\mathbb{Z}$?

   (a) $\{\{x : x < 0\}, \{0\}, \{x : x > 0\}\}$
   (b) $\{\{x : x \text{ is even }\}, \{x : x \text{ is odd }\}\}$
   (c) $\{\{x : x \text{ is divisible by 3 }\}, \{x : x \text{ is not divisible by 3 }\}\}$
   (d) $\{\{x : \text{the last digit of } x \text{ is in } \{1, 2, 3, 4\}\}, \{x : \text{the last digit of } x \text{ is in } \{6, 7, 8, 9\}, \{x : 5 \mid x\}\}$
   (e) $\{\{x : x \leq 0\}, \{x : x \geq 0\}\}$
   (f) $\{\{x : x < 0\}, \{x : x > 0\}\}$

   Answer: all but the last two

3. How many partitions are there of the following sets?

   (a) $\{1, 2\}$
   (b) $\{2, 4\}$
   (c) $\{\mathbb{Z}, \mathbb{N}\}$
   (d) $\{\{\emptyset\}, \emptyset\}$
   (e) $\{1\}$
   (f) $\{\mathbb{Z}\}$
   (g) $\{1, 2, 3\}$
   (h) $\{a, b, c\}$

   Answers: 2,2,2,2,1,1,5,5

4. How many partitions P are there of the set $\{1, 2, 3, 4, 5, 6\}$ that meet the following criteria?

   (a) $\{1, 2\}$ and $\{2, 3\}$ are in P

(b) $\{1,2\}$ and $\{3,4\}$ are in P

(c) $\{1,2,3\}$ and $\{4\}$ are in P

(d) $\{1,2,3,4\}$ is in P

(e) $\{1,2,3,5,6\}$ is in P

Answers: 0,2,2,2,1

5. What partition of $\{1,2,3,4,5,6\}$ do we get for each of the following equivalence classes?

   (a) aRb iff a and b have the same parity.

   (b) aRb iff $3|a-b$.

   (c) aRb iff $5|a-b$.

   (d) aRb iff ab is positive.

   (e) aRb iff $a-3.5$ and $b-3.5$ are either both positive or both negative real numbers

   (f) aRb iff $|a|=|b|$.

   Answers:

   (a) $\{[1],[2]\} \quad (=\{\{1,3,5\},\{2,4,6\}\})$

   (b) $\{[1],[2],[3]\} \quad (=\{\{1,4\},\{2,5\},\{3,6\}\})$

   (c) $\{[1],[2],[3],[4],[5]\} \quad (=\{\{1,6\},\{2\},\{3\},\{4\},\{5\}\})$

   (d) $\{[1]\} \quad (=\{\{1,2,3,4,5,6\}\})$

   (e) $\{[1],[4]\} \quad (=\{\{1,2,3\},\{4,5,6\}\})$

   (f) $\{[1],[2],[3],[4],[5],[6]\} \quad (=\{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\})$

6. How many distinct equivalence classes are there in the partition generated by the following equivalence relation.

   (a) $A=\{1,2,3,4,5,6,7\}$ and aRb iff 6 divides $b-a$.

   (b) $A=\{1,2,3,4,5,6,7\}$ and aRb iff 4 divides $b-a$.

   (c) $A=\{1,2,3,4,5,6,7\}$ and aRb iff $b+a$ is even.

   Answers: 6,4,2

Chapter 6

Functions

6.1 Functions

1. Which of the following relations on $A = \{1, 2, 3, 4, 5, 6\}$ are functions?

   (a) $R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 1)\}$ [Answer: Function]
   (b) $R = \{(1, 2), (2, 1), (3, 4), (4, 3), (5, 6), (6, 5)\}$ [Answer: Function]
   (c) $R = \{(1, 2), (2, 2), (3, 4), (4, 4), (5, 6), (6, 6)\}$ [Answer: Function]
   (d) $R = \{(1, 1), (1, 2), (3, 3), (3, 4), (5, 5), (5, 6)\}$ [Answer: Not a Function]
   (e) $R = \emptyset$ [Answer: Not a Function]
   (f) $R = \{(x, y) \in A \times A : x < y\}$ [Answer: Not a Function]
   (g) $R = \{(x, y) \in A \times A : x = y + 1\}$ [Answer: Not a Function]
   (h) $R = \{(x, y) \in A \times A : x = 6 - y\}$ [Answer: Not a Function]
   (i) $R = \{(x, y) \in A \times A : x = 7 - y\}$ [Answer: Function]
   (j) $R = \{(x, y) \in A \times A : x = 8 - y\}$ [Answer: Not a Function]
   (k) $R = \{(x, y) \in A \times A : x = y\}$ [Answer: Function]
   (l) $R = \{(x, y) \in A \times A : y = 2\}$ [Answer: Function]
   (m) $R = \{(x, y) \in A \times A : x = 2\}$ [Answer: Not a Function]
   (n) $R = \{(x, y) \in A \times A : xy = 6\}$ [Answer: Not a Function]
   (o) $R = A \times A$ [Answer: Not a Function]

2. Which of the following relations on $A \times B = \{1, 2, 3\} \times \{4, 5, 6\}$ are functions?

   (a) $R = \{(1, 4), (1, 5), (1, 6)\}$ [Answer: Not a Function]
   (b) $R = \{(1, 5), (2, 5)\}$ [Answer: Not a Function]
   (c) $R = \{(1, 5), (2, 5), (3, 5)\}$ [Answer: Function]
   (d) $R = \{(1, 5), (2, 5), (3, 5), (3, 6)\}$ [Answer: Not a Function]

(e) $R = \{(x, y) \in A \times B : y = x + 3\}$ [Answer: Function]
(f) $R = \{(x, y) \in A \times B : y = 7 - x\}$ [Answer: Function]
(g) $R = \{(x, y) \in A \times B : y = 5\}$ [Answer: Function]
(h) $R = \{(x, y) \in A \times B : x = 2\}$ [Answer: Not a Function]

3. Which of the following relations on $\mathbb{N}$ are functions?

(a) $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} : y = x + 3\}$ [Answer: Function]
(b) $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} : y = x - 3\}$ [Answer: Not a Function]
(c) $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x^2 = y\}$ [Answer: Function]
(d) $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x = y^2\}$ [Answer: Not a Function]
(e) $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} : xy = x\}$ [Answer: Function]
(f) $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} : xy = y\}$ [Answer: Not a Function]

4. Which of the following relations on $\mathbb{Z}$ are functions?

(a) $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : y = x + 3\}$ [Answer: Function]
(b) $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : y = x - 3\}$ [Answer: Function]
(c) $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x^2 = y\}$ [Answer: Function]
(d) $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x = y^2\}$ [Answer: Not a Function]
(e) $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : xy = x\}$ [Answer: Not a Function]
(f) $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : xy = y\}$ [Answer: Not a Function]

5. Which of the following relations on $\mathbb{R}$ are functions?

(a) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = 0\}$ [Answer: Function]
(b) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = 0\}$ [Answer: Not a Function]
(c) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y + x = 0\}$ [Answer: Function]
(d) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x + 3\}$ [Answer: Function]
(e) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 = y\}$ [Answer: Function]
(f) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = y^2\}$ [Answer: Not a Function]
(g) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |y| = |x|\}$ [Answer: Not a Function]
(h) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : xy = x\}$ [Answer: Not Function]
(i) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : xy = y\}$ [Answer: Not a Function]
(j) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : xy = 1\}$ [Answer: Not a Function]
(k) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : (1 + x)y = 1\}$ [Answer: Not a Function]
(l) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : (1 + x^2)y = 1\}$ [Answer: Function]

6. For each of the following A and B state the total number of functions from A to B and, when possible, give an example of a function from A to B.

(a) $A = \{1, 2\}, B = \{3\}$ [Answer: 1, $R = \{(1, 3), (2, 3)\}$]

(b) $A = \{1\}, B = \{2, 3\}$ [Answer: 2, $R = \{(1, 2)\}$]

(c) $A = \{1, 2\}, B = \{3, 4\}$ [Answer: 4, $R = \{(1, 3), (2, 4)\}$]

(d) $A = \{1, 2\}, B = \{6, 7\}$ [Answer: 4, $R = \{(1, 6), (2, 7)\}$]

(e) $A = \{1, 2\}, B = \emptyset$ [Answer: 0]

(f) $A = \emptyset, B = \{1, 2\}$ [Answer: 1, $R = \emptyset$]

(g) $A = \{\emptyset, \{\emptyset\}\}, B = \{\emptyset\}$ [Answer: 1, $R = \{(\emptyset, \emptyset), (\{\emptyset\}, \emptyset)\}$]

(h) $A = \{\emptyset\}, B = \{\emptyset, \{\emptyset\}\}$ [Answer: 2, $R = \{(\emptyset, \emptyset)\}$]

(i) $A = \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}$ [Answer: 4, $R = \{(\emptyset, \emptyset), (\{\emptyset\}, \emptyset)\}$]

6.2 Images and Inverse Images

For each of the following functions, find the images/inverse images requested. In these problems we set $\mathbb{E}$ to be the even integers, $\mathbb{O}$ to be the odd integers, A to be the set $\{1, 2, 3\}$, B to be the set $\{-1, 0, 1\}$, and $\mathbb{P}$ to be the primes.

1. $f(n) : \mathbb{Z} \to \mathbb{Z}$ where $f(n) = 2n$

   (a) $f(A)$
   (b) $f(B)$
   (c) $f(\mathbb{E})$
   (d) $f(\mathbb{O})$
   (e) $f^{-1}(A)$
   (f) $f^{-1}(B)$
   (g) $f^{-1}(\mathbb{E})$
   (h) $f^{-1}(\mathbb{O})$
   (i) $f^{-1}(\mathbb{P})$

   Solution:

   (a) $\{2, 4, 6\}$
   (b) $\{-2, 0, 2\}$
   (c) $\{\cdots, -8, -4, 0, 4, 8, \cdots\} = \{4k : k \in \mathbb{Z}\} = \{k \in \mathbb{Z} : 4 \mid k\}$
   (d) $\{\cdots, -10, -6, -2, 2, 6, 10, \cdots\} = \{4k + 2 : k \in \mathbb{Z}\} = \{k \in \mathbb{Z} : 4 \mid (k - 2)\} = \{k \in \mathbb{Z} : 4 \mid (k + 2)\}$
   (e) $\{1\}$
   (f) $\{0\}$
   (g) $\mathbb{Z}$
   (h) $\emptyset$
   (i) $\{1\}$

2. $f(n) : \mathbb{Z} \to \mathbb{Z}$ where $f(n) = n + 5$

   (a) $f(A)$
   (b) $f(B)$
   (c) $f(\mathbb{E})$
   (d) $f(\mathbb{O})$
   (e) $f^{-1}(A)$
   (f) $f^{-1}(B)$
   (g) $f^{-1}(\mathbb{E})$
   (h) $f^{-1}(\mathbb{O})$

Solution:

(a) $\{6, 7, 8\}$
(b) $\{4, 5, 6\}$
(c) $\mathbb{O}$
(d) $\mathbb{E}$
(e) $\{-4, -3, -2\}$
(f) $\{-6, -5, -4\}$
(g) $\mathbb{O}$
(h) $\mathbb{E}$

3. $f(n) : \mathbb{Z} \to \mathbb{Z}$ where $f(n) = 2n + 1$

   (a) $f(A)$
   (b) $f(B)$
   (c) $f(\mathbb{E})$
   (d) $f(\mathbb{O})$
   (e) $f^{-1}(A)$
   (f) $f^{-1}(B)$
   (g) $f^{-1}(\mathbb{E})$
   (h) $f^{-1}(\mathbb{O})$

4. $f(n) : \mathbb{Z} \to \mathbb{Z}$ where $f(n) = 3n$

   (a) $f(A)$
   (b) $f(B)$
   (c) $f(\mathbb{E})$
   (d) $f(\mathbb{O})$
   (e) $f^{-1}(A)$
   (f) $f^{-1}(B)$
   (g) $f^{-1}(\mathbb{E})$
   (h) $f^{-1}(\mathbb{O})$

5. $f((m, n)) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $f((m, n)) = mn$

   (a) $f(\mathbb{E} \times \mathbb{E})$
   (b) $f(\mathbb{O} \times \mathbb{O})$
   (c) $f(\mathbb{O} \times \mathbb{E})$
   (d) $f^{-1}(A)$

(e) $f^{-1}(\mathbb{E})$
(f) $f^{-1}(\mathbb{O})$
(g) $f^{-1}(\mathbb{P})$

6. $f(n) : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where $f(n) = (n, n+1)$

   (a) $f(A)$
   (b) $f(B)$
   (c) $f(\mathbb{E})$
   (d) $f^{-1}(\mathbb{E} \times \mathbb{E})$
   (e) $f^{-1}(\mathbb{E} \times \mathbb{O})$
   (f) $f^{-1}(\mathbb{O} \times \mathbb{E})$

7. $f(n) : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where $f(n) = (n+2, n-2)$

   (a) $f(A)$
   (b) $f(B)$
   (c) $f(\mathbb{E})$
   (d) $f^{-1}(\mathbb{E} \times \mathbb{E})$
   (e) $f^{-1}(\mathbb{O} \times \mathbb{O})$
   (f) $f^{-1}(\mathbb{E} \times \mathbb{O})$
   (g) $f^{-1}(\mathbb{O} \times \mathbb{E})$

   Solution:

   (a) $\{(3, -1), (4, 0), (5, 1)\}$
   (b) $\{(1, -3), (2, -2), (3, -1)\}$
   (c) $\{(2k + 2, 2k - 2) : k \in \mathbb{Z}\} = \{(2k, 2k - 4) : k \in \mathbb{Z}\} = \{(2k + 4, 2k) : k \in \mathbb{Z}\}$
   (d) $\mathbb{E}$
   (e) $\mathbb{O}$
   (f) $\emptyset$
   (g) $\emptyset$

8. $f((m, n)) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where $f((m, n)) = (1, n)$

   (a) $f(A \times A)$
   (b) $f(B \times B)$
   (c) $f^{-1}(\mathbb{E} \times \mathbb{E})$
   (d) $f^{-1}(\mathbb{O} \times \mathbb{O})$
   (e) $f^{-1}(\mathbb{E} \times \mathbb{O})$

(f) $f^{-1}(\mathbb{O} \times \mathbb{E})$

Solution:

(a) $\{(1,1),(1,2),(1,3)\}$
(b) $\{(1,-1),(1,0),(1,1)\}$
(c) $\emptyset$
(d) $\mathbb{Z} \times \mathbb{O}$
(e) $\emptyset$
(f) $\mathbb{Z} \times \mathbb{E}$

9. $f((m,n)) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where $f((m,n)) = (n,m)$

(a) $f(A \times A)$
(b) $f(B \times B)$
(c) $f^{-1}(\mathbb{E} \times \mathbb{E})$
(d) $f^{-1}(\mathbb{O} \times \mathbb{O})$
(e) $f^{-1}(\mathbb{E} \times \mathbb{O})$
(f) $f^{-1}(\mathbb{O} \times \mathbb{E})$

Solution:

(a) $A \times A$
(b) $B \times B$
(c) $\mathbb{E} \times \mathbb{E}$
(d) $\mathbb{O} \times \mathbb{O}$
(e) $\mathbb{O} \times \mathbb{E}$
(f) $\mathbb{O} \times \mathbb{E}$

10. $f((m,n)) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where $f((m,n)) = (m+n,0)$

(a) $f(A \times A)$
(b) $f(B \times B)$
(c) $f^{-1}(\mathbb{E} \times \mathbb{E})$
(d) $f^{-1}(\mathbb{E} \times \mathbb{O})$
(e) $f^{-1}(\mathbb{O} \times \mathbb{E})$

6.3 Injectivity and Surjectivity

1. State whether the function f is injective, surjective, bijective, or none of the above.

   (a) $A=\{1,2,3\}, B=\{1,2,3\},\ f:A\to B,\ f=\{(a,b)\in A\times A : a=4-b\}$ [Answer: Bijective.]

   (b) $A=\{1,2,3\}, B=\{1,2,3\},\ f:A\to B,\ f=\{(a,b)\in A\times A : b\equiv a+1 \pmod 3\}$ [Answer: Bijective.]

   (c) $A=\{1,2,3\}, B=\{4,5,6\},\ f:A\to B,\ f=\{(1,6),(2,5),(3,4)\}$ [Answer: Bijective.]

   (d) $A=\{1,2,3\}, B=\{4,5,6\},\ f:A\to B,\ f=\{(1,6),(2,5),(3,5)\}$ [Answer: Neither.]

   (e) $A=\{1,2,3,4,5\}, B=\{6,7,8,9,10\},\ f:A\to B,\ f=\{(1,6),(2,7),(3,8),(4,9),(5,10)\}$ [Answer: Bijective.]

   (f) $A=\{1,2,3,4,5\}, B=\{6,7,8,9,10\},\ f:A\to B,\ f=\{(1,8),(2,7),(3,6),(4,9),(5,10)\}$ [Answer: Bijective.]

   (g) $A=\{1,2,3,4,5\}, B=\{6,7,8,9,10\},\ f:A\to B,\ f=\{(1,8),(2,7),(3,8),(4,7),(5,8)\}$ [Answer: Neither.]

   (h) $A=\{1,2,3,4,5\}, B=\{7,8\},\ f:A\to B,\ f=\{(1,8),(2,7),(3,8),(4,7),(5,8)\}$ [Answer: Only surjective.]

   (i) $A=\{1,2\}, B=\{6,7,8,9,10\},\ f:A\to B,\ f=\{(1,8),(2,7)\}$ [Answer: Only injective.]

   (j) $A=\{1,2\}, B=\{3,4,5\}, C=\{6,7,8,9\},\ h:A\to B,\ h=\{(1,3),(2,4)\}, g:B\to C,\ g=\{(3,6),(4,7),(5,8)\}, f=g\circ h$. [Answer: Only injective.]

   (k) $A=\{1,2,3\}, B=\{4,5,6\}, C=\{7,8\},\ h:A\to B,\ h=\{(1,4),(2,5),(3,6)\}, g:B\to C,\ g=\{(4,7),(5,8),(6,7)\}, f=g\circ h$. [Answer: Only surjective.]

   (l) $A=\{1,2\}, B=\{3,4,5\}, C=\{6,7,8,9\},\ h:A\to B,\ h=\{(1,3),(2,4)\}, g:B\to C,\ g=\{(3,6),(4,7),(5,6)\}, f=g\circ h$. [Answer: Neither.]

   (m) $A=\{1,2\}, B=\{3,4\}, C=\{5,6\}, h:A\to B, h=\{(1,3),(2,4)\}, g:B\to C, g=\{(3,5),(4,6)\}, f=g\circ h$. [Answer: Both.]

   (n) $A=\{1,2\}, B=\{3,4\}, C=\{5,6\}, h:A\to B, h=\{(1,4),(2,3)\}, g:B\to C, g=\{(3,5),(4,6)\}, f=g\circ h$. [Answer: Both.]

   (o) $A=\{1,2\}, B=\{3,4\}, C=\{5,6\}, h:A\to B, h=\{(1,4),(2,3)\}, g:B\to C, g=\{(3,5),(4,5)\}, f=g\circ h$. [Answer: Neither.]

2. State whether the following functions are injective, surjective, bijective, or none of the above.

   (a) $f:\{2,3\}\to\{4,5,6,7,8,9\},\ f(x)=x^2$ [Answer: Only injective.]

   (b) $f:\{2,3\}\to\{4,9\},\ f(x)=x^2$ [Answer: Bijective.]

   (c) $f:\{-3,-2,2,3\}\to\{4,9\},\ f(x)=x^2$ [Answer: Only surjective.]

   (d) $f:\mathbb{Z}\to\{-1,1\},\ f(x)=(-1)^n$ [Answer: Only surjective.]

   (e) $f:\{1,2\}\to\{-1,1\},\ f(x)=(-1)^n$ [Answer: Bijective.]

   (f) $f:\{1,3\}\to\{-1,1\},\ f(x)=(-1)^n$ [Answer: Neither.]

   (g) $f:\{1\}\to\{-1\},\ f(x)=(-1)^n$ [Answer: Bijective.]

(h) $f : \mathbb{Q} \to \mathbb{R}$, $f(x) = x$ [Answer: Only injective.]
(i) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ [Answer: Neither.]
(j) $f : \mathbb{R} \to [0, \infty)$, $f(x) = x^2$ [Answer: Only surjective.]
(k) $f : [0, 1.] \to \mathbb{R}$, $f(x) = x^2$ [Answer: Only injective.]
(l) $f : [0, 1.] \to [0, 1.]$, $f(x) = x^2$ [Answer: Bijective.]
(m) $f : (0, 1) \to (0, 1)$, $f(x) = x^2$ [Answer: Bijective.]
(n) $f : [0, \infty) \to [0, \infty)$, $f(x) = x^2$ [Answer: Bijective.]
(o) $f : [0, \infty) \to [0, \infty)$, $f(x) = |x|$ [Answer: Bijective.]
(p) $f : [-1, 1.] \to [-1, 1.]$, $f(x) = |x|$ [Answer: Neither.]
(q) $f : [-1, 1.] \to [0, 1.]$, $f(x) = |x|$ [Answer: Only surjective.]
(r) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ [Answer: Neither.]
(s) $f : \mathbb{R} \to [-1, 1.]$, $f(x) = \sin(x)$ [Answer: Only surjective.]
(t) $f : [0, \pi/2.] \to \mathbb{R}$, $f(x) = \sin(x)$ [Answer: Only injective.]
(u) $f : [0, \pi/2.] \to [0, 1.]$, $f(x) = \sin(x)$ [Answer: Bijective.]
(v) $f : (0, \infty) \to \mathbb{R}$, $f(x) = \ln(x)$ [Answer: Bijective.]
(w) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$ [Answer: Only injective.]
(x) $f : \mathbb{R} \to (0, \infty)$, $f(x) = e^x$ [Answer: Bijective.]
(y) $f : (0, \infty) \to (1, \infty)$, $f(x) = 2^x$ [Answer: Bijective.]
(z) $f : (0, \infty) \to [1, \infty)$, $f(x) = 2^x$ [Answer: Only injective.]

3. State how many injective, surjective and bijective functions exist from A to B.

(a) $A = \{1\}$, $B = \{1\}$ [Answer: 1,1,1.]
(b) $A = \{1, 2\}$, $B = \{1\}$ [Answer: 0,1,0.]
(c) $A = \{1\}$, $B = \{1, 2\}$ [Answer: 2,0,0.]
(d) $A = \{1, 2, 3\}, B = \{1\}$ [Answer: 0,1,0.]
(e) $A = \{1\}, B = \{1, 2, 3\}$ [Answer: 3,0,0.]
(f) $A = \{1, 2\}, B = \{1, 2\}$ [Answer: 2,2,2.]
(g) $A = \{1, 2, 3\}, B = \{1, 2, 3\}$ [Answer: 6,6,6.]
(h) $A = \{1, 2, 3, 4\}, B = \{1, 2, 3, 4\}$ [Answer: 24,24,24.]
(i) $A = \{1, 2, 3, 4, 5\}, B = \{1, 2, 3, 4, 5\}$ [Answer: 120,120,120.]
(j) $A = \{1, 2, 3, 4, 5, 6\}, B = \{1, 2, 3, 4, 5, 6\}$ [Answer: 720,720,720.]
(k) $A = \{1, 2\}, B = \{1, 2, 3\}$ [Answer: 6,0,0.]
(l) $A = \{1, 2\}, B = \{1, 2, 3, 4\}$ [Answer: 12,0,0.]
(m) $A = \{1, 2\}, B = \{1, 2, 3, 4, 5\}$ [Answer: 20,0,0.]
(n) $A = \{1, 2\}, B = \{1, 2, 3, 4, 5, 6\}$ [Answer: 30,0,0.]
(o) $A = \{1, 2, 3\}, B = \{1, 2\}$ [Answer: 0,6,0.]
(p) $A = \{1, 2, 3, 4\}, B = \{1, 2\}$ [Answer: 0,14,0.]
(q) $A = \{1, 2, 3, 4, 5\}, B = \{1, 2\}$ [Answer: 0,30,0.]
(r) $A = \{1, 2, 3, 4, 5, 6\}, B = \{1, 2\}$ [Answer: 0,62,0.]

6.4 Proofs With Injectivity and Surjectivity

For each of the following functions, find out if they are injective and find out if they are surjective. Prove each of your conclusions is correct.

1. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(n) = n + 1$
2. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(n) = 2n + 1$
3. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(n) = 2n$
4. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(n) = 3n - 5$

   Solution: f is injective but not surjective.

   Proof: f is injective. Suppose $f(a) = f(b)$. We will show that $a = b$. $f(a) = f(b)$ implies $3a-5 = 3b-5$. Adding 5 to both sides tells us $3a = 3b$ and dividing by three gives us $a = b$.

   f is not surjective. There is no $a \in \mathbb{Z}$ so that $f(a) = 0$ as this would demand $3a - 5 = 0$ which implies $3a = 5$ and $a = 5/3 \notin \mathbb{Z}$. □

5. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(n) = 2 - n$
6. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(n) = 2 - 3n$

   Solution: f is injective but not surjective.

   Proof: f is injective. If $f(a) = f(b)$ then $2 - 3a = 2 - 3b$. Subtracting two from both sides gives us $-3a = -3b$ and multiplying by negative three shows that $a = b$.

   f is not surjective. $f(n) \neq 0$ for any $n \in \mathbb{Z}$. To see this notice that if $f(n) = 0$ then $2 - 3n = 0$ so $2 = 3n$ which is impossible in $\mathbb{Z}$. □

7. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(n) = 1 - 2n$
8. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(n) = |n|$
9. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(n) = n^2 + 2$

   Solution: f is neither injective nor surjective.

   f is not injective. $g(1) = 3 = g(-1)$.

   f is not surjective. $f(n) \neq 0$ for any $n \in \mathbb{Z}$. To see this notice that if $f(n) = 0$ then $n^2 + 1 = 0$ so $n^2 = -1$ which is impossible in $\mathbb{Z}$. □

10. $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = n + 1$
11. $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = |n|$
12. $f : \mathbb{E} \to \mathbb{E}$ where $f(n) = 2 - n$
13. $f : \mathbb{E} \to \mathbb{E}$ where $f(n) = 2n$

14. $f : \mathbb{Z} \to \mathbb{E}$ where $f(n) = 2n - 6$

    Solution: f is both injective and surjective.

    Proof: f is injective. Suppose $f(a) = f(b)$. We will show that $a = b$. $f(a) = f(b)$ implies $2a-6 = 2b-6$. Adding 5 to both sides tells us $2a = 2b$ and dividing by two gives us $a = b$.

    f is surjective. Let b be any even number in $\mathbb{E}$. We must find $a \in \mathbb{Z}$ so $f(a) = b$. We are looking for a so $2a - 6 = b$ so we know $2a = b + 6$. As both b and 6 are even $b + 6$ is an even number so $\frac{1}{2}(b + 6)$ is an integer. Simply let $a = \frac{1}{2}(b + 6)$ and we are done. □

15. $f : \mathbb{E} \to \mathbb{Z}$ where $f(n) = n/2 + 1$

16. $f : \mathbb{E} \to \mathbb{E} \cup \{0\}$ where $f(n) = |n/2|$

    Solution: f is surjective but not injective.

    Proof: f is not injective since $f(2) = f(-2)$.

    f is surjective. Let b be any nonnegative integer. We want an a so $|a/2| = b$. If we pick nonnegative a then $|a/2| = a/2$ and we only need $a/2 = b$. Simply let $a = 2b$. Then a is nonnegative so our conclusion is met. □

17. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $f((m, n)) = mn$

18. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $f((m, n)) = mn^2$

19. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $f((m, n)) = 2mn$

20. $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where $f(n) = (n, n + 1)$

21. $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where $f(n) = (n^2, -n^2)$

22. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where $f((m, n)) = (n, m)$

23. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where $f(m, n) = (n - 1, m + n)$

    Solution: f is both injective and surjective.

    Proof: f is injective. Suppose $f(a, b) = f(c, d)$. We will show that $(a, b) = (c, d)$. $f(a, b) = f(c, d)$ implies $(b - 1, a + b) = (d - 1, c + d)$ so $b - 1 = d - 1$ and $a + b = c + d$. Since $b - 1 = d - 1$ we know $b = d$ so $a + b = c + b$. This implies $a = c$. Since $b = d$ and $a = c$ we know $(a, b) = (c, d)$.

    f is surjective. Suppose (c, d) is any point in $\mathbb{Z}\times\mathbb{Z}$. We will show we can find (a, b) so that $f(a, b) = (c, d)$. $f(a, b) = (b-1, a+b)$ so we need to find (a, b) so that $b-1 = c$ and $a+b = d$. We can set $b = c+1$ and $a = d-b = d-(c+1) = d-c-1$. Then $f(d-c-1, c+1) = (c+1-1, d-c-1+c+1) = (c, d)$. □

24. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where $f((m, n)) = (m + n, m - n)$

25. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where $f((m, n)) = (m + n, 0)$

26. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where $f(x, y) = (x^2 + y, x)$

    Solution: f is both injective and surjective.

    Proof: f is injective. If $f(a, b) = f(c, d)$ then $(a^2 + b, a) = (c^2 + d, c)$ so $a^2 + b = c^2 + d$ and $a = c$. Thus $c^2 + b = c^2 + d$ and $b = d$. Since $a = c$ and $b = d$ we know $(a, b) = (c, d)$.

f is surjective. Suppose (c,d) is any point in $\mathbb{Z}\times\mathbb{Z}$. We want to find (a,b) in $\mathbb{Z}\times\mathbb{Z}$ so that $f(a,b)=(c,d)$. This means $(a^2+b,a)=(c,d)$ so $a=d$ and $a^2+b=c$. We can set $a=d$ and then want $d^2+b=c$ or $b=c-d^2$. Then $f(a,b)=(a^2+b,a)=(d^2+c-d^2,d)=(c,d)$. $\square$

27. $f:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}\times\mathbb{Z}$ where $f(x,y)=((x+y)^2,x)$

    Solution: f is neither injective nor surjective.

    Proof: f is not injective. If $f(a,b)=f(c,d)$ then $((a+b)^2,a)=((c+d)^2,c)$ so $(a+b)^2=(c+d)^2$ and $a=c$. Thus $(a+b)^2=(a+d)^2$ and $(a+b)=\pm(a+d)$ If $(a+b)=(a+d)$ then $b=d$ and we would be fine, but $a+b$ could equal $-(a+d)$. There $a+b=-a-d$ so we can seek to set $d=-2a-b$. Consider $(a,b)=(1,1)$ and $(c,d)=(1,-3)$. Then note $f(1,1)=(4,1)$ and $f(1,-3)=(4,1)$ yet $(1,1)\neq(1,-3)$.

    f is not surjective. There is no (a,b) so $f(a,b)=(-1,0)$. $\square$

28. $f:\mathbb{R}\to\mathbb{R}$ where $f(x)=1-2x$

29. $f:\mathbb{R}\to\mathbb{R}$ where $f(x)=3x-5$

    Solution: f is both injective and surjective.

    Proof: f is injective. Suppose $f(a)=f(b)$. We will show that $a=b$. $f(a)=f(b)$ implies $3a-5=3b-5$. Adding 5 and dividing by 3 gives us $a=b$.

    f is surjective. Let b be any real number. We want an a so $f(a)=b$ or $3a-5=b$. Simply set $a=\frac{1}{3}(b+5)$. $\square$

30. $f:\mathbb{R}\to\mathbb{R}$ where $f(x)=3-3x^3$

31. $f:\mathbb{R}\to\mathbb{R}$ where $f(x)=a^2-2a+5$

    Solution: f is neither injective nor surjective.

    Proof: f is not injective since $f(2)=f(0)$.

    f is not surjective since there is no $a\in\mathbb{R}$ so $f(a)=0$. The quadratic equation shows $x^2-2x+5=0$ has no real solutions. $\square$

32. $f:[0,\infty)\to\mathbb{R}$ where $f(x)=1+\sqrt{x}$

    Solution: f is injective but not surjective.

    Proof: f is injective. Suppose $f(a)=f(b)$. We will show that $a=b$. $f(a)=f(b)$ implies $1+\sqrt{a}=1+\sqrt{b}$. Subtracting 1 to both sides gives $\sqrt{a}=\sqrt{b}$ and squaring both sides gives us $b=a$.

    f is not surjective since there is no $a\in[0,\infty)$ so $f(a)=0$. We would need $\sqrt{x}=-1$ which is not possible in $\mathbb{R}$. $\square$

33. $f:\mathbb{R}-\{1\}\to\mathbb{R}-\{1\}$ where $f(x)=\frac{x+1}{x-1}$

    Solution: f is both injective and surjective.

    Proof: f is injective. Suppose $f(a)=f(b)$. We will show that $a=b$. $f(a)=f(b)$ implies $\frac{a+1}{a-1}=\frac{b+1}{b-1}$. Multiplying both sides by $(a-1)(b-1)$ gives $(a+1)(b-1)=(b+1)(a-1)$ so $ab+b-a-1=ba-b+a-1$. Subtract $ab-1$ from both sides to get $b-a=a-b$. This implies $2b=2a$ and thus $b=a$.

    f is surjective. Let b be any real number not equal to 1. We want an a so $\frac{a+1}{a-1}=b$ or $a+1=b(a-1)$. This means $ba-b-a-1=0$ or $a(b-1)=b+1$. Since $b\neq 1$ we can divide to get $a=\frac{b+1}{b-1}$ which works as our a. $\square$

6.5 Bijections and Cardinality

1. Prove that the following sets have the same cardinality by finding a bijection from one to the other. You do not have to prove your function is a bijection, but it must be one.

   (a) $[0,1]$ and $[-2,0]$
   (b) $[0,\pi/2]$ and $[0,1]$ [Answer: $f:[0,\pi/2]\to[0,1], f(x)=\sin(x)$ or $f(x)=\cos x$ or $f(x)=\frac{2}{\pi}x$.]
   (c) $[-1,1]$ and $[0,\pi]$
   (d) $(0,1)$ and $(0,2)$ [Answer: $f:(0,1)\to(0,2), f(x)=2x$ or $f(x)=2\sqrt{x}$ or $f(x)=\sqrt{4x}$.]
   (e) $(0,1)$ and $(3,5)$ [Answer: $f:(0,1)\to(3,5), f(x)=2x+3$.]
   (f) $(1,3)$ and $(-3,7)$
   (g) $(1,4]$ and $[0,3)$[Answer: $f:(1,4]\to[0,3), f(x)=4-x$.]
   (h) $(1,4]$ and $[0,1)$[Answer: $f:(1,4]\to[0,1), f(x)=\frac{1}{3}(4-x)$.]
   (i) $(1,4]$ and $[1,2)$ [Answer: $f:(1,4]\to[1,2), f(x)=\frac{1}{3}(4-x)+1$.]
   (j) $[0,1)$ and $(0,1]$
   (k) $[0,1)$ and $(-3,4]$ [Answer: $f:[0,1)\to(-3,4], f(x)=-7x+4$.]
   (l) $[0,4)$ and $(-3,1]$
   (m) $[0,\infty)$ and $[1,\infty)$ [Answer: $f:[0,\infty)\to[1,\infty), f(x)=x+1$ or $f(x)=e^x$ or $f(x)=x^2+1$.]
   (n) $(0,\infty)$ and (π,∞) [Answer: $f:(0,\infty)\to(\pi,\infty), f(x)=e^x+\pi-1$.
   (o) $(0,\infty)$ and $(-\infty,2)$
   (p) $(-\pi/2,\pi/2)$ and $\mathbb{R}$ [Answer: $f:(-\pi/2,\pi/2)\to\mathbb{R}, f(x)=\tan(x)$.]
   (q) $\mathbb{R}$ and $(0,\pi)$ [Answer: $f:\mathbb{R}\to(0,\pi), f(x)=\arctan(x)+\pi$.]
   (r) $\mathbb{R}$ and $(0,1)$ [Answer: $f:\mathbb{R}\to(0,1), f(x)=\frac{\arctan(x)+\pi}{\pi}$.]
   (s) $\mathbb{R}$ and $(1,\infty)$
   (t) $\mathbb{N}$ and $\{1,4,9,16,\cdots\}$
   (u) $\{5,10,15,20,\cdots\}$ and $\{-2,-1,0,1,2,3,\cdots\}$
   (v) $\mathbb{E}\cap\mathbb{N}$ and $\{3,6,9,12,15,\cdots\}$
   (w) $\mathbb{N}\cap\mathbb{E}$ and $\mathbb{N}-\mathbb{E}$
   (x) $\{\cdots,-2,-1,0,1,2,3\}$ and $\{-2,-1,0,1,\cdots\}$
   (y) $\{-3,-2,-1,0,1,2,\cdots\}$ and $\{\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},\cdots\}$.
   (z) $\{\cdots,-11,-8,-5,-2\}$ and $\{4,6,8,10,\cdots\}$ [Answer: $f:\{\cdots,-11,-8,-5,-2\}\to\{4,6,8,10,\cdots\}$, $f(x)=4-\frac{2}{3}(x+2)$.]

2. For each of the following sets, prove that they are denumerable [1]. If you state that something is a bijection, try to prove that it really is.

   (a) $\{-5,-3,-1,1,3\}$[Answer: $f:\mathbb{N}\to\{-5,-3,-1,1,3\}, f(x)=5-2x$.]

[1] This simply means they have the same cardinality as $\mathbb{N}$.

(b) $\mathbb{E} - \mathbb{N}$ [Answer: $f : \mathbb{N} \to \mathbb{E} - \mathbb{N}, f(x) = -2x + 2$.]

(c) $\mathbb{E} \cap \mathbb{N}$

(d) $\mathbb{E} - \mathbb{N}$

(e) $(\mathbb{Z} - \mathbb{E}) \cap \mathbb{N}$

Solution: We want to find a bijection between $\mathbb{N}$ and $(\mathbb{Z} - \mathbb{E}) \cap \mathbb{N}$. The first set can be written $\{1, 2, 3, 4, 5, \cdots\}$ and the second can be written $\{1, 3, 5, 7, \cdots\}$. Lets try to find an $f(x)$ sending 1 to 1, 2 to 3, 3 to 5 and so on. Notice that in the image of the function we want, the entries differ by 2, so lets try multiplying by 2. The function $f(x) = 2x$ doesn't work, since it sends 1 to 2, 2 to 4 and so on, but we can correct this by subtracting 1 from each of the numbers. This will give us $f(x) = 2x - 1$. We now prove this is a bijection.

Proof: For injectivity note that if $f(a) = f(b)$ then $2a - 1 = 2b - 1$. Adding 1 gives us $2a = 2b$. Dividing by 2 allows us to conclude that $a = b$.

For surjectivity note that if b is in $(\mathbb{Z}-\mathbb{E})\cap\mathbb{N}$ then b is of the form $2k+1$ (from $(\mathbb{Z}-\mathbb{E})$) and k must be greater than or equal to 0 (from $\mathbb{N}$.) We must find an a in $\mathbb{N}$ so that $f(a)$ equals this $2k + 1$. Let's try $k+1$. Since $k \geq 0$, $k+1$ must be in $\mathbb{N}$. Also $f(k+1) = 2(k+1)-1 = 2k+2-1 = 2k+1$ which is exactly what we wanted to show. $\square$

(f) $(\mathbb{Z} - \mathbb{E}) - \mathbb{N}$

Solution: We want to find a bijection between $\mathbb{N}$ and $(\mathbb{Z} - \mathbb{E}) - \mathbb{N}$. The first set can be written $\{1, 2, 3, 4, 5, \cdots\}$ and the second can be written $\{-1, -3, -5, \cdots\}$. Lets try to find an $f(x)$ sending 1 to -1, 2 to -3, 3 to -5 and so on. Notice that in the image of the function we want, we need the entries to differ by 2, so lets try multiplying by 2 and taking the negatives. The function $f(x) = -2x$ doesn't work, since it sends 1 to -2, 2 to -4 and so on, but we can correct this by adding 1 to each of the numbers. This will give us $f(x) = -2x+1$. We now prove this is bijective.

Proof: We still need to prove this is a bijection. For injectivity note that if $f(a) = f(b)$ then $-2a+1 = -2b+1$. Subtracting 1 gives us $-2a = -2b$. Dividing by -2 allows us to conclude that $a = b$.

For surjectivity note that if b is in $(\mathbb{Z} - \mathbb{E}) \cap \mathbb{N}$ then b is of the form $2k + 1$ (from $(\mathbb{Z} - \mathbb{E})$) and k must be less than zero. We must find an a in $\mathbb{N}$ so that $f(a)$ equals this $2k+1$. If $f(a) = b = 2k+1$ then $-2a + 1 = 2k + 1$ so $-2a = 2k$ and thus a must equal $-k$. Since $k < 0$ we know $a > 0$ which is exactly what we needed since $a \in \mathbb{N}$. So picking $a = -k$ gives us an a in $\mathbb{N}$ where $f(a) = b$. $\square$

(g) $\{\cdots, -7, -6, -5, -4\}$

(h) $\{\cdots, -9, -7, -5, -3\}$

(i) $\{\cdots, -12, -7, -2, 3\}$

(j) $\{-8, -5, -2, 1, 4, \cdots\}$

(k) $\{\sqrt{3}, \sqrt{4}, \sqrt{5}, \cdots\}$

(l) $\{\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{8}}\}$

Solution: We want to find a bijection between $\{\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{8}}\}$. The first set can be written $\{1, 2, 3, 4, 5, \cdots\}$ Lets try to find an $f(x)$ sending 1 to $\frac{1}{\sqrt{5}}$, 2 to $\frac{1}{\sqrt{6}}$, 3 to $\frac{1}{\sqrt{7}}$, and so on. We need radicals in the denominator so we could try $f(x) = \frac{1}{\sqrt{x}}$ which takes the reciprocal of the square root. This doesn't work, however, because it would send $1, 2, 3, 4, \cdots$ to $\frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, \cdots$. We need the entries in the radical to be four more than they are, so we can try $f(x) = \frac{1}{\sqrt{x+4}}$. Lets show that this works by proving this is a bijection.

Proof: For injectivity note that if $f(a) = f(b)$ then $\frac{1}{\sqrt{a+4}} = \frac{1}{\sqrt{b+4}}$. This implies $\sqrt{a+4} = \sqrt{b+4}$ and squaring both sides tells us that $a + 4 = b + 4$. Subtracting four allows us to conclude that $a = b$.

For surjectivity note that if b is in $\{\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{8}}\}$ then it equals $\frac{1}{\sqrt{k}}$ for some integer $k \geq 5$. We need to find the a so $f(a) = b = \frac{1}{\sqrt{k}}$ and show that a is a natural number. $\frac{1}{\sqrt{a+4}} = \frac{1}{\sqrt{k}}$ if and only if $\sqrt{a+4} = \sqrt{k}$ which is true exactly when $a + 4 = k$. since $k \geq 5$ we know $a > 0$. Since k is an integer, a is also an integer. As a positive integer, it is in the natural numbers and we have shown our $f(x)$ to be surjective. $\square$

(m) $\{\frac{3}{5}, \frac{3}{6}, \frac{3}{7}, \frac{3}{8}, \cdots\}$

(n) $\{\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \cdots\}$

(o) $\{\frac{8}{7}, -\frac{9}{7}, \frac{10}{7}, -\frac{11}{7}, \cdots\}$

(p) $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots\}$

(q) $\{\frac{2}{1}, \frac{3}{8}, \frac{4}{27}, \frac{5}{64}, \cdots\}$

(r) $\{2, 5, 10, 17, 26, 37, \cdots\}$

Appendix A

Definitions

A.1 Statements

Definition A.1.1. *A statement or proposition is a sentence that is definitely either true or false.*

Definition A.1.2. *A logical operator is a symbol used to connect propositions to form a new sentence whose truth value depends only on the values of those proposition in some way.*

Definition A.1.3. *The truth table for a statement Q involving propositions P_1, P_2, $\cdots$, P_n is a table containing every possible arrangement of truth values for our P_i together with whether or not Q is true or false under those conditions.*

Definition A.1.4. *The negation of a statement P is the new statement, written $\sim P$, which is considered true if and only if P is not true.*

Definition A.1.5. *The conjunction of two statements P and Q is the new statement, written $P \wedge Q$, which is considered true if and only if both P and Q are true.*

Definition A.1.6. *The disjunction of two statements P and Q is the new statement, written $P \vee Q$, which is considered true if at least one of P and Q are true.*

Definition A.1.7. *Given statements P and Q, the conditional, written $P \Rightarrow Q$, is the statement "if P then Q" which is always considered true except for when P is true and Q is not.*

Definition A.1.8. *The biconditional of two statements P and Q is the new statement, written $P \Leftrightarrow Q$, which is considered true if P and Q have the same truth value.*

Definition A.1.9. *Two statements Q and R, made from the propositions $P_1, P_2, \cdots, P_n$ are logically equivalent if they are both true under the exact same conditions of the truth values of our P_i. In this case we write $P \equiv R$.*

Definition A.1.10. *A statement made from the propositions $P_1, P_2, \cdots, P_n$ is a tautology if it is true for every possible assignment of truth values for our P_i.*

Definition A.1.11. *A statement made from the propositions $P_1, P_2, \cdots, P_n$ is a contradiction if it is false for every possible assignment of truth values for our P_i.*

Definition A.1.12. *The contrapositive of the conditional statement $P \Rightarrow Q$ is the statement $\sim Q \Rightarrow \sim P$.*

Definition A.1.13. *The universe of discourse for is the collection of objects currently being discussed.*

Definition A.1.14. *A quantified statement $P(n)$ is a statement containing a variable that is either true or false for each n in our universe of discourse.*

Definition A.1.15. *A universal quantifier $\forall n$ when placed before a quantified statement $P(n)$, forms a new statement which is true if and only if $P(n)$ is true for each n in our universe of discourse.*

Definition A.1.16. *An existential quantifier $\exists n$ when placed before a quantified statement $P(n)$, forms a new statement which is true if and only if $P(n)$ is true for some n in our universe of discourse.*

A.2 Sets

Definition A.2.1. *A set is a collection of objects.*

Definition A.2.2. *The cardinality of a set is the number of elements in that set.*

Definition A.2.3. *The empty set $\emptyset$ is the collection of no objects.*

Definition A.2.4. *Two sets are equal if they contain the same elements.*

Definition A.2.5. *We call a an element of the set A, and write $a \in A$, if a is one of the objects in the set A.*

Definition A.2.6. *We call the set A a subset of the set B and write $A \subseteq B$, if every element in A is also in B.*

Definition A.2.7. *The power set of the set A, is the collection $\mathcal{P}(\mathcal{A})$ of all subsets of A.*

Definition A.2.8. *The union $A \cup B$ of the sets A and B, is the collection of all objects that are in A or B.*

Definition A.2.9. *The intersection $A \cap B$ of the sets A and B, is the collection of all objects that are in A and B.*

Definition A.2.10. *Given two sets A and B, the set A setminus B, written $A - B$, is the collection of all the objects in A that are not in B.*

Definition A.2.11. *When considering sets as all subsets of some universe of discourse U, the complementA^C of the set A is the collection $U - A$.*

Definition A.2.12. *The cartesian product of the sets A and B, is the collection $A \times B$ of ordered pairs (a, b) where $a \in A$ and $b \in B$.*

Definition A.2.13. *A set is infinite if it contains infinitely many objects. Otherwise it is called finite.*

Definition A.2.14. *The natural numbers $\mathbb{N}$ are the set $\{1, 2, 3, \cdots\}$ of all positive whole numbers.*

Definition A.2.15. *The integers $\mathbb{Z}$ are the set $\{\cdots - 3, -2, -1, 0, 1, 2, 3, \cdots\}$ of all whole numbers.*

Definition A.2.16. *The rational numbers $\mathbb{Q}$ are the set $\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\}$.*

Definition A.2.17 (Not a real definition)**.** *The real numbers $\mathbb{R}$ are the (surprisingly difficult to define) set of numbers familiar to math majors from use in courses such as College Algebra, Pre-calculus, and Calculus.*

Definition A.2.18. *The irrational numbers are the collection of all real numbers that are not rational.*

Definition A.2.19. *The complex numbers $\mathbb{C}$ are the set $\{a + bi : a \in \mathbb{R}, b \in \mathbb{R}\}$.*

A.3 Number Properties and Sequences

Definition A.3.1. *An integer n is even if it is of the form $2k$ for some k in $\mathbb{Z}$.*

Definition A.3.2. *An integer n is odd if it is of the form $2k+1$ for some k in $\mathbb{Z}$.*

Definition A.3.3. *We say that two integers have the same parity if they are both even or both odd.*

Definition A.3.4. *We say a non-zero integer b divides the integer a and write $b|a$ if $a = bk$ for some $k \in \mathbb{Z}$.*

Definition A.3.5. *The prime numbers are the set of all positive integers with exactly two positive divisors.*

Definition A.3.6. *We say that a is equivalent to b modulo n and write $a \equiv b \pmod{n}$ if n divides $b - a$.*

Definition A.3.7. *Given two real numbers x and y, we say that x is less than y and write $x < y$ if $y - x$ is positive.*

Definition A.3.8. *The absolute value of a real number x is defined to be $-x$ if x is negative, and x if it is not.*

Definition A.3.9. *A sequence is an ordered collection of (often infinitely many) objects from some set.*

Definition A.3.10. *A recurrence relation is a sequence where each term is defined by a formula involving the preceding terms.*

A.4 Relations and Functions

Definition A.4.1. *A relation on* $A \times B$ *is a subset* R *of* $A \times B$*. When* $B = A$ *we say that* R *is a relation on* A*.*

Definition A.4.2. *The domain of a relation* R *on* $A \times B$ *is the set dom*$(R) = \{a \in A : (a, b) \in R$ *for some* $b \in B.\}$

Definition A.4.3. *The range of a relation* R *on* $A \times B$ *is the set ran*$(R) = \{b \in B : (a, b) \in R$ *for some* $a \in A.\}$

Definition A.4.4. *The inverse of a relation* R *on* $A \times B$ *is the set* $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R.\}$

Definition A.4.5. *Given a relation* R *on* $A \times B$ *and a relation* S *on* $B \times C$ *the composition of* S *and* R*, written* $S \circ R$*, is* $\{(a, c) :$ *there is some* $b \in B$ *so that* $(a, b) \in R$ *and* $(b, c) \in S.\}$

Definition A.4.6. *A relation* R *on* A *is reflexive if* $(a, a) \in R$ *for every* $a \in A$*.*

Definition A.4.7. *A relation* R *on* A *is symmetric if whenever* $(a, b) \in R$ *we have* $(b, a) \in R$*.*

Definition A.4.8. *A relation* R *on* A *is transitive if whenever* $(a, b) \in R$ *and* $(b, c) \in R$ *we have* $(a, c) \in R$*.*

Definition A.4.9. *A relation* R *on* A *is reflexive, symmetric and transitive is called an equivalence relation.*

Definition A.4.10. *The equivalence class of* a *in an equivalence relation on* A *is the set* $[a] = \{b \in A : (a, b) \in R.\}$

Definition A.4.11. *A partition of a set* A *is a collection* P *of nonempty subsets of* A *so that*

1. *Every* $x \in A$ *is in some set* $B \in P$*.*
2. *If* B *and* C *are in* P *then either* $B \cap C = \emptyset$ *or* $B = C$*.*

Definition A.4.12. *A relation on* $A \times B$ *is a function from* A *to* B *if*

1. *dom*$(R) = A$*.*
2. *If* (x, y) *and* (x, z) *are in* R *then* $y = z$*.*

If a relation is a function we write $f(a) = b$ *for* $(a, b) \in R$*. We also write* $f : A \to B$ *to indicate that* f *is a function from* A *to* B*.*

Definition A.4.13. *The image of the set* A *under the function* f *on* $A \times B$ *is the set* $f(A) = \{b \in B : \exists a \in A$ *so* $f(a) = b\}$*. This equals the set of all possible outputs of elements of* A*.*

Definition A.4.14. *The pre image of the set* B *under the function* f *on* $A \times B$ *is the set* $f^{-1}(B) = \{a \in A : f(a) \in B\}$*. This equals the set of all elements in* A *that have outputs in* B*. Note that despite the notation,* f *does not need to be invertible for us to consider this set.*

Definition A.4.15. *A function on* $A \times B$ *is injective if* $f(a) = f(b)$ *implies* $a = b$ *for every* a *and* b *in* A*. This is equivalent to saying no two elements of* A *are sent by* f *to the same element. This is also sometimes given in the contrapositive form of* $a \neq b$ *implies* $f(a) \neq f(b)$*.*

Definition A.4.16. *A function on* $A \times B$ *is surjective if for every* $b \in B$ *there is some* $a \in A$ *so that* $f(a) = b$*.*

Definition A.4.17. *A function on* $A \times B$ *is bijective if it is both injective and surjective.*

Definition A.4.18. *A set* A *is countably infinite or denumerable if there is a bijection from* $\mathbb{N}$ *to* A*.*

Appendix B

Counting Formulas

Assuming all sets are finite, the following formulas hold.

1. The number of subsets of A, also known as $|\mathcal{P}(A)|$ is equal to $2^{|A|}$.
2. The size of the cartesian product of A and B is $|A| \times |B|$.
3. The number of relations on $A \times B$ is $2^{|A| \times |B|}$.
4. The number of functions from A to B is $|B|^{|A|}$.
5. The number of bijective functions from A to B is $|A|!$ if $|A| = |B|$ and zero otherwise.
6. The number of injective functions from A to B is $\frac{|B|!}{(|B|-|A|)!}$ if $|A| \leq |B|$ and zero otherwise.

Appendix C

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